4 Conjectures involving 2 of Richard's finest ideas

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Graph $G = (V, E)$

Proper vertex coloring κ: adjacent vertices get different colors.

$x_1^2 x_3$

$x_1 x_2 x_3$

Chromatic symmetric function [Richard, 1995]:

$$X_G(x_1, x_2, \ldots) = X_G(\mathbf{x}) = \sum_{\text{proper } \kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)}.$$

Can $X_G(\mathbf{x})$ distinguish graphs?

$$X_G(\mathbf{x}) = \sum_{\text{proper } \kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_n)}.$$

Richard: these have the same $X_G(\mathbf{x})$:

Famous Statement [Richard].
"We do not know whether X_G distinguishes trees."
i.e. if T and U are non-isomorphic trees, then is $X_T(\mathbf{x}) \neq X_U(\mathbf{x})$?

[Aliniaeifard, Aliste-Prieto, Crew, Dahhberg, de Mier, Fougere, Heil, Ji, Loebl, Loehr, Martin, Morin, Orellana, Scott, Smith, Sereni, Spirkl, Tian, Wagner, Wang, Warrington, van Willigenburg, Zamora, ...]

Conjecture 1 [Richard]. $X_G(\mathbf{x})$ distinguishes trees. In other words, if T and U are non-isomorphic trees, then $X_T(\mathbf{x}) \neq X_U(\mathbf{x})$.

The Loehr–Warrington Conjecture

What if we plug in values for the x_i?

Principial specialization (up to a constant power of q):
Set $x_i = q^i$ for $1 \le i \le n$;
Set $x_i = 0$ for $i > n$.

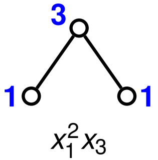
Now $X_G(q, q^2, q^3, \ldots, q^n)$ is a polynomial in a single variable so contains much less information than $X_G(x_1, x_2, \ldots)$.

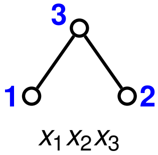
(Surprising) Conjecture 2 [Nick Loehr & Greg Warrington, 2022].
$X_G(q, q^2, q^3, \ldots, q^n)$ also distinguishes trees with n vertices.

- The data suggests that fewer than n nonzero variables suffice.
- See also [Bajo, Beck, Vindas-Meléndez, 2024]

Another fine idea in 1971: P-partitions

strict P-partition
Any poset P
$f : P \to \{1, 2, 3, \ldots\}$ that is strictly order-preserving





Strict P-partition enumerator:

$$K_P(x_1, x_2, \ldots) = K_P(\mathbf{x}) = \sum_f x_1^{\# f^{-1}(1)} x_2^{\# f^{-1}(2)} x_3^{\# f^{-1}(3)} \cdots.$$

Project. Study equality among $K_P(\mathbf{x})$.

[Albertin, Aval, Browning, Djenabou, Féray, Hasebe, Hopkins, Kelly, Lesnevich, Liu, M., Mlodecki, Tsujie, Ward, Weselcouch, ...]

Can $K_P(\mathbf{x})$ distinguish posets?

Parallel story....

Conjecture 3 [Jean-Christophe Aval, Karimatou Djenabou, M., 2022; stated as a question by Takahiro Hasebe and Shuhei Tsujie, 2017].

$K_P(\mathbf{x})$ distinguishes posets whose Hasse diagrams are trees.

Notes.

- An affirmative answer to Conjecture 3 would prove that the chromatic quasisymmetric function for digraphs [Elllzey, Shareshian, Wachs] distinguishes directed trees.
- All conjectures here have substantial numerical evidence.

One more!

Conjecture 4 [Aval–Djenabou–M., 2022].
$K_P(q, q^2, q^3, \ldots, q^n)$ also distinguishes n-element posets that are trees.

Remark. This specialization has a nice interpretation: if

$$K_P(q, q^2, q^3, \ldots, q^n) = \sum_{i \geq 0} a(i)\, q^i,$$

then $a(i)$ counts the number of strict P-partitions $f : P \to [n]$ of i.

Theorems and further details in [Aval, Djenabou, M., *Quasisymmetric functions distinguishing trees*, 2022].