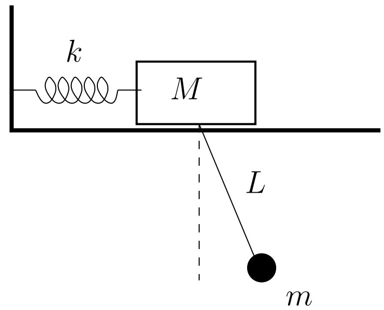
PHYS 331 — Exam #3

Wednesday November 15, 2017

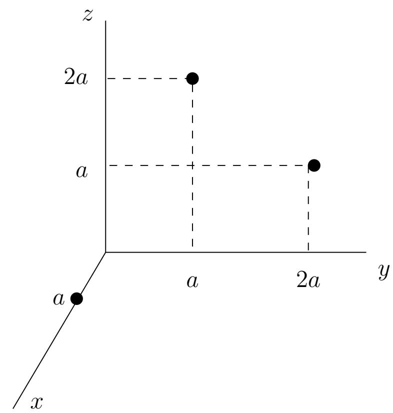
1. (25 pts) A simple pendulum with mass m and length L is suspended from a frictionless cart with mass M that is attached to a spring with force constant k as illustrated in the figure.



   (a) Find the Lagrangian for the system in the limit of small oscillations of the cart and pendulum. Give your answer in terms the given variables and physical constants.

   (b) Let the numerical values of the variables (in some system of units) be such that $m = M = g = L = 1$, and $k = 2$. Find the frequencies of the normal modes and describe the motion in the normal modes.

2. (25 pts) A rigid body consists of three equal masses (m) fastened at the illustrated positions by massless connectors.



The positions of the masses can be written

$$\begin{aligned} \mathbf{r}_1 &= a\,\hat{x} + 0\,\hat{y} + 0\,\hat{z}, \\ \mathbf{r}_2 &= 0\,\hat{x} + a\,\hat{y} + 2a\,\hat{z}, \\ \mathbf{r}_3 &= 0\,\hat{x} + 2a\,\hat{y} + a\,\hat{z}. \end{aligned}$$

(a) Determine the inertia tensor $\mathbf{I}$ about the origin for this distribution of masses.

(b) Assume that the rigid body has an angular velocity $\boldsymbol{\omega} = 5\,\hat{z}$. Determine the angular momentum of the body about the origin .

(c) Use your inertia tensor to find an axis about which the angular momentum vector will be proportional to the angular velocity vector and determine the constant of proportionality.

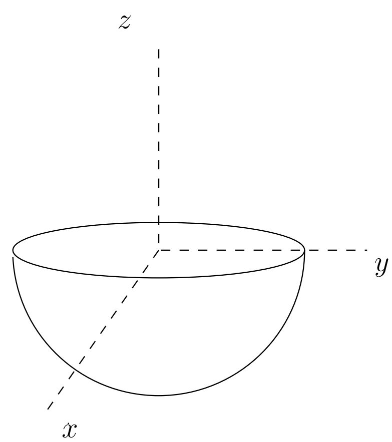
3. (20 pts) In Lewisburg (co-latitude $\theta = 49°$) a bullet with a mass $0.002\,\text{kg}$ is fired straight up into the air with a speed of $350\,\text{m/s}$.

   (a) Determine the magnitude and direction of the centrifugal force on the bullet.

   (b) Determine the magnitude and direction of the Coriolis force on the bullet.

   HINT and NOTE: Diagrams may be helpful here. Be clear about your directions — you might use diagrams, unit vectors in an appropriate (and well-specified) coordinate system, or articulate use of language, or some combination of these.

4. (15 pts) Find the position of the center of mass of the illustrated uniform solid hemisphere of radius R and mass M, with the flat side of the hemisphere lying in the x-y plane, and the rest of the hemisphere below the plane.



5. (15 pts) A frame $\mathcal{S}$ rotates with a constant angular velocity $\mathbf{\Omega}$ with respect to an inertial frame $\mathcal{S}_0$. Show that the equation of motion for a particle moving in the rotating frame S is

$$m\ddot{\mathbf{r}} = \mathbf{F} + 2m\dot{\mathbf{r}}\times\mathbf{\Omega} + m(\mathbf{\Omega} \times \mathbf{r})\times\mathbf{\Omega},$$

where $\mathbf{F}$ is the net force on the body as determined in the inertial frame $\mathcal{S}_0$, and the "dots" indicate time derivatives evaluated with respect to the rotating frame $\mathcal{S}$.

Useful Information

$$\mathbf{F}_{\text{inertial}} = \mathbf{F}_{\text{fictitious}} = -m\mathbf{A}$$

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

$$f(x) = f(x_0) + (x - x_0)\left.\frac{df}{dx}\right|_{x=x_0} + \frac{1}{2}(x - x_0)^2 \left.\frac{d^2 f}{dx^2}\right|_{x=x_0} + \cdots$$

$$\left(\frac{d\mathbf{Q}}{dt}\right)_{\mathcal{S}_0} = \left(\frac{d\mathbf{Q}}{dt}\right)_{\mathcal{S}} + \boldsymbol{\Omega} \times \mathbf{Q}$$

$$\mathbf{F}_{\text{cor}} = 2m\dot{\mathbf{r}} \times \boldsymbol{\Omega} \quad \text{and} \quad \mathbf{F}_{\text{cf}} = m(\boldsymbol{\Omega} \times \mathbf{r})\times\boldsymbol{\Omega}$$

$$\mathbf{L} = \mathbf{L}(\text{motion of CM}) + \mathbf{L}(\text{motion relative to CM})$$

$$T = T(\text{motion of CM}) + T(\text{motion relative to CM})$$

$$\mathbf{L} = \mathbf{I}\boldsymbol{\omega}$$

$$I_{xx} = \sum_{\alpha} m_\alpha (y_\alpha^2 + z_\alpha^2), \text{etc.} \quad \text{and} \quad I_{xy} = -\sum_{\alpha} m_\alpha x_\alpha y_\alpha, \text{etc.}$$

$$\text{Principal Axis:} \quad \mathbf{L} = \lambda\boldsymbol{\omega}$$

$$\dot{\mathbf{L}} + \boldsymbol{\omega}\times\mathbf{L} = \boldsymbol{\Gamma}$$

$$T = \frac{1}{2}\sum_{j,k} M_{jk}\dot{q}_j\dot{q}_k \quad \text{and} \quad U = \frac{1}{2}\sum_{j,k} K_{jk} q_j q_k$$

$$(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$$