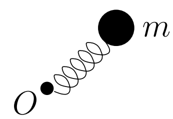
1. (35 pts) A mass m moves on a horizontal two-dimensional frictionless surface. A spring connects the mass to a fixed pivot point at the origin. The spring constant is k and the equilibrium length of the spring is zero. (This is a two-dimensional version of a neutral atom trap, in which the “spring” force is the magnetic force that acts on the magnetic dipole moment of the neutral atom.)

Top view



   (a) Determine the lagrangian $\mathcal{L}$ as a function of the polar coordinates r, ϕ of the mass.

   (b) From your lagrangian, determine the equations of motion for r and ϕ.

   (c) Show that the angular momentum of the mass is conserved, and rewrite your equations of motion in terms of the angular momentum l.

   (d) Determine the equilibrium radial length r_0 (in terms of m, l, and k) and describe the motion of the mass for $r - r_0$.

   (e) Now imagine that the mass is perturbed slightly from r_0. Show that the equilibrium is stable, and determine the frequency of small oscillations about the equilibrium radial length r_0.

2. (15 pts) Consider a system of 3 particles in 3D. The only forces are conservative interactions between the particle, i.e., there are no external forces. Show that translational invariance leads to conservation of the x-component of the **total** momentum, i.e.,

$$P_x = p_{1x} + p_{2x} + p_{3x} = \text{constant}.$$

3. (20 pts) The orbit of Comet Encke has an eccentricity of 0.847, and its closest approach to the sun is 0.339 AU, where 1 AU is the distance from the earth to the sun.

   (a) Determine the semi-major axis of the orbit of Comet Encke in AU.

   (b) Determine the period of the orbit of Comet Encke. Remember that you know the period of a solar system object with semi-major axis of 1 AU! (You may take advantage of the fact that $M_{\text{sun}} \gg m_{\text{comet}}$).

4. (10 pts) For each of the following scenarios, determine

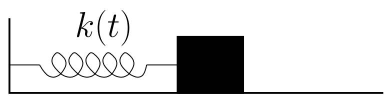
   (i) whether the Hamiltonian $\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L}$ is conserved (yes or no)

   (ii) whether energy is conserved (yes or no)

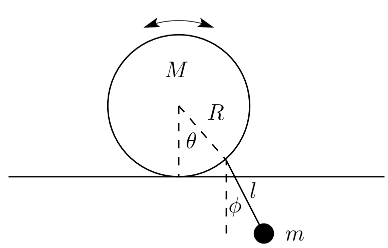
   (iii) whether $\mathcal{H}$ gives the energy (yes or no)

   Give your reasoning.

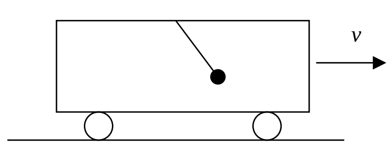
   (a) A frictionless mass-spring system with a time-dependent spring constant, $k = k_0 + \alpha t$, where α is a constant.



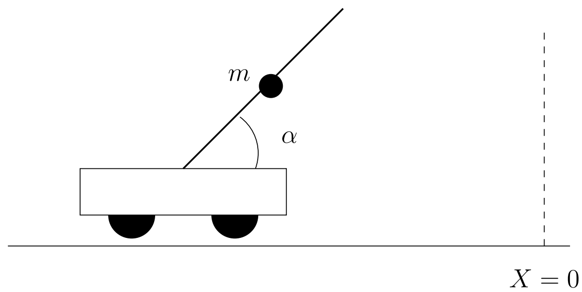
   (b) A pendulum attached to the rim of a uniform disk that is free to roll back and forth.



   (c) A pendulum inside a train car that moves with constant velocity. (Careful — this one is a bit tricky.)



5. (20 pts) **Note:** It is straightforward to solve this problem without using lagrangians, but I want you to demonstrate that you know how to use the lagrangian method here.



Consider a bead of mass m sliding without friction on a rigid straight wire that is inclined at a fixed angle α above the horizontal x-axis, in the presence of a gravitational field pointing down the page. The base of the wire is attached to a cart that is maintained with a constant acceleration to the left (by an external force), so that the position of the center of the cart is

$$X = -\frac{1}{2}at^2.$$

(a) Define any necessary generalized coordinates and find the lagrangian for the system.

(b) Find the Euler-Lagrange equation (or equations) for this system.

(c) Solve the resulting equation(s) of motion.

Useful Information

$$\mathcal{L} = \mathcal{L}(q_i, \dot{q}_i, t)$$

$$\mathcal{L} = T - U$$

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

$$\mathcal{H} = \sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L}$$

$$r(\phi) = \frac{c}{1 + \epsilon \cos\phi}$$

$$E = \frac{\gamma^2 \mu}{2\ell^2}(\epsilon^2 - 1)$$

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \quad \text{and} \quad \mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m m_2}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$U_{\text{eff}} = U(r) + \frac{\ell^2}{2\mu r^2}$$

$$f(x) = f(x_0) + (x - x_0)\left.\frac{df}{dx}\right|_{x=x_0} + \frac{1}{2}(x - x_0)^2 \left.\frac{d^2 f}{dx^2}\right|_{x=x_0} + \cdots$$