Topic 6

RC Circuits & Filters — the *Frequency Domain*

Material discussed 2/5/19, 2/7/19, 2/12/19

6.1 Introduction

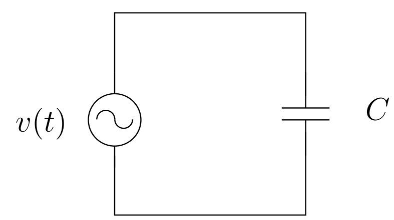
Previously we have dealt with the response of RC circuits to step-function changes in voltages. Here we look at the response to continuous AC sinusoidal voltage signals.

6.1.1 Resistors

For DC voltages and currents there is a direct proportionality between the voltage across a resistor and the current through the resistor, and the same proportionality (Ohm's Law) holds for ideal resistors with AC voltages and currents:

$$\frac{v_R(t)}{i_R(t)} = \text{real constant} \longrightarrow R,$$

6.1.2 Capacitors



Things are a little more complicated when we consider the voltage across a capacitor and the instantaneous current flowing onto the plates of a capacitor. For a capacitor we have

$$v_C = \frac{q}{C},$$

which gives

$$q = CV_C,$$

and the current flowing onto the capacitor is

$$i_C = \frac{dq}{dt} = C\frac{dv_C}{dt}.$$

In the simple illustrated circuit $v_C = v_s = A\cos(\omega t + \phi)$, so

$$\begin{aligned} i_C &= -\omega CA\sin(\omega t + \phi) \\ &= \omega CA\cos(\omega t + \phi + \pi/2). \end{aligned}$$

Notice that the voltage v_C and the current i_C are not related by a simple multiplicative constant like they are in a resistor. The *amplitudes* of the voltage and the current are related by the multiplicative constant ωC, but there is a *phase difference* between $v_C(t)$ and $i_C(t)$ of $\pi/2$, or 90°. The voltage across the resistor (determined by the current) *leads* the voltage across the capacitor by a phase of $\pi/2$, or 90°. The ratio of the amplitudes of the voltage and the current is

$$\frac{|v_C|}{|i_C|} = \frac{1}{\omega C}.$$

This factor $1/\omega C$ is a resistance-like quantity which we will soon see is the *magnitude* of the complex *impedance*. In a nutshell, for "high" frequencies the capacitive impedance is small, and for "low" frequencies the capacitive impedance is large; capacitors *impede* low frequency signals, and don't impede high frequency signals.

Complex impedance. The complex number representation gives us a convenient way to deal with phase differences that arise in AC circuits with capacitors, and we'll show that for capacitors

$$\frac{\tilde{v}_C}{\tilde{i}} = \text{frequency-dependent imaginary number} \longrightarrow Z_C,$$

and we call the imaginary constant the *impedance* of the capacitor, Z_C.

To determine Z_C, consider the complex form our AC voltage and current:

$$\tilde{v}_C = Ae^{j(\omega t+\phi)},$$

and

$$\tilde{i}_C = \omega C A e^{j(\omega t+\phi+\pi/2)}.$$

The ratio of these is

$$\frac{\tilde{v}_C}{\tilde{i}_C} = \frac{1}{\omega C e^{j\pi/2}} = \frac{1}{j\omega C} = -\frac{j}{\omega C}.$$

This complex ratio is the impedance of the capacitor, or

$$Z_C = \frac{1}{j\omega C} = -\frac{j}{\omega C}.$$

Low frequencies imply high capacitive impedances and vice-versa.

[You may also see the capacitive impedance written as

$$Z_C = -jX_C$$

where $X_C = 1/\omega C$ is the magnitude of the real factor in the impedance; X_C is called the *capacitive reactance*. But beware: in some references the capacitive reactance is defined as $X_C = -1/\omega C$, so $Z_C = +jX_C$.]

6.2 Generalized Ohm's law

Complex form of Ohm's law:

$$\tilde{i} = \frac{\tilde{v}}{Z} \qquad \text{or} \qquad \tilde{v} = \tilde{i}Z,$$

where Z is the *complex impedance*. As mentioned above, for resistors Ohm's law works the same way for AC as it does for DC, so we have

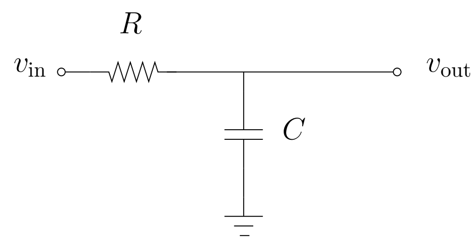
$$Z_R = R$$

and for a capacitor we have

$$Z_C = \frac{1}{j\omega C} = -\frac{j}{\omega C}.$$

Series and parallel combinations of Z's combine with the same rules used for resistances.

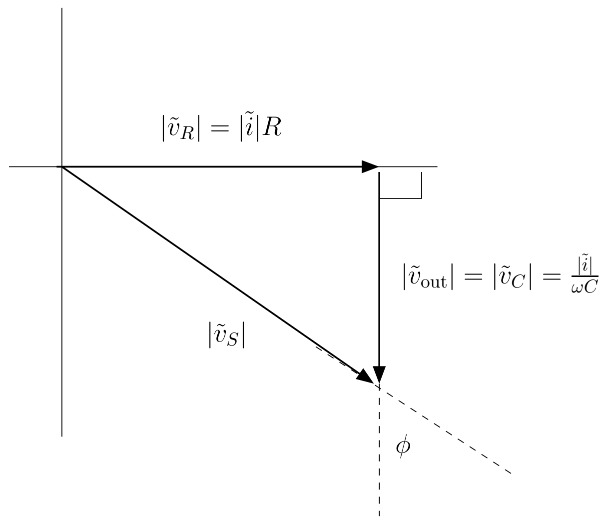
6.3 Low-pass RC filter



6.3.1 Qualitative discussion

- The RC low-pass filter can be considered as a generalized voltage divider. (Remember that the output of a resistive voltage divider is given by $v_{\text{in}} \times R_2/(R_1 + R_2)$.) In the generalized voltage divider the capacitor plays the role of R_2.
- At low frequencies the *impedance* of the capacitor is large, which is analogous to having a large R_2 in a resistive voltage divider. The output voltage will be approach the input voltage as the frequency approaches zero.
- At high frequencies the *impedance* of the capacitor is small, which is analogous to having a small R_2 in a resistive voltage divider. The output voltage will be approach zero as the frequency gets large.
- Similar arguments can be applied to a high-pass filter.
- This qualitative discussion does not deal with the phase of the output relative to the input.

6.3.2 Quantitative with phasors



- Voltage across the resistor leads voltage across the capacitor by $\pi/2$.
- KVL:

$$\tilde{v}_S = \tilde{v}_R + \tilde{v}_C$$

- Geometry/trig gives same results as those given in next section. Use KVL to find amplitude of current:

$$\begin{aligned} |\tilde{v}_S|^2 &= |\tilde{i}|^2 R^2 + |\tilde{i}|^2 \frac{1}{(\omega C)^2} \\ \Longrightarrow \qquad |\tilde{i}| &= \frac{|\tilde{v}_S|}{\sqrt{R^2 + \frac{1}{(\omega C)^2}}}. \end{aligned}$$

Using this gives

$$\begin{aligned} |\tilde{v}_{\text{out}}| = |\tilde{v}_C| &= |\tilde{i}] \frac{1}{\omega C} \\ &= \frac{1}{\omega C} \frac{1}{\sqrt{R^2 + \frac{1}{(\omega C)^2}}} \\ &= \frac{1}{\sqrt{1 + (R\omega C)^2}} |\tilde{v}_{\text{in}}|. \end{aligned}$$

From the figure, the phase angle between $\tilde{v}_S$ and $\tilde{v}_{\text{out}}$ is

$$\phi = -\tan^{-1}\left(\frac{R}{\frac{1}{\omega C}}\right) = -\tan^{-1}(\omega RC).$$

The minus sign comes from the fact that the output voltage lags the input voltage.

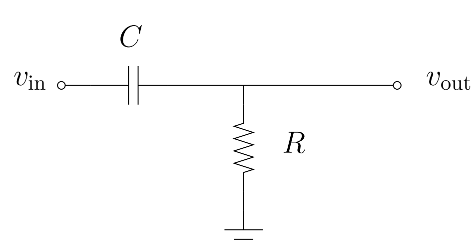
6.3.3 Quantitative with complex numbers

Treat the low-pass filter as a complex voltage divider:

$$\begin{aligned}
\tilde{v}_{\text{out}} &= \frac{Z_C}{Z_R + Z_C}\,\tilde{v}_{\text{in}} \\
&= \frac{-\frac{j}{\omega C}}{R - \frac{j}{\omega C}}\,\tilde{v}_{\text{in}} \\
&= \frac{1}{1 + j\omega RC}\,\tilde{v}_{\text{in}} \\
&= \frac{1}{\sqrt{1 + (\omega RC)^2}}\,e^{j\phi_{\text{low}}}\,\tilde{v}_{\text{in}}
\end{aligned}$$

where $\phi_{\text{low}} = -\tan^{-1}(\omega RC)$.

6.4 High-pass filter



Treat this as a complex voltage divider:

$$
\begin{aligned}
\frac{\tilde{v}_{\text{out}}}{\tilde{v}_{\text{in}}} &= \frac{Z_R}{Z_R + Z_C} \\
&= \frac{R}{R - \frac{j}{\omega C}} \\
&= \frac{R\omega C}{R\omega C - j} \\
&= \frac{\omega RC}{\sqrt{1 + (\omega RC)^2}} e^{j\phi_{\text{high}}}
\end{aligned}
$$

where $\phi_{\text{high}} = \tan^{-1}(1/\omega RC)$.

6.5 Inductors

[Not originally discussed in class, but added for completeness in 2/12 lecture.]

6.5.1 Introduction to inductors

The complex number representation gives us a convenient way to deal with phase differences that arise in AC circuits with inductors. As for capacitors, we find that for inductors

$$\frac{\tilde{v}_L}{\tilde{i}_L} = \text{frequency-dependent imaginary number},$$

and we call the imaginary number the *impedance* of the inductor. For inductors

$$\frac{\tilde{v}_L}{\tilde{i}_L} = j\omega L,$$

where L is the inductance measured in henries (H). This complex number is the impedance of the inductor, or

$$Z_L = j\omega L.$$

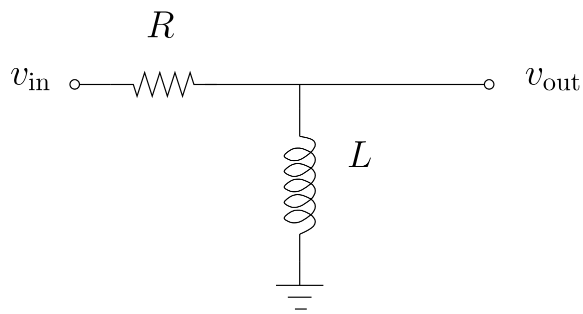
High frequencies imply high inductive impedances and vice-versa.

[You may also see the inductive impedance written as

$$Z_L = jX_L$$

where $X_L = \omega L$ is the real factor in the impedance; X_L is called the *inductive reactance*.]

6.5.2 Example with inductors



Qualitative: Consider this as a generalized voltage divider. Low frequency $\Longrightarrow$ low inductive impedance $\Longrightarrow$ small voltage across inductor $\Longrightarrow$ small gain. Conversely, high frequency $\Longrightarrow$ high inductive impedance $\Longrightarrow$ gain approaches unity. This is a high-pass filter.

Quantitative: Again, consider this as a voltage divider.

$$\begin{aligned}
\frac{\tilde{v}_{\text{out}}}{\tilde{v}_{\text{in}}} &= \frac{Z_L}{Z_L + Z_R} \\
&= \frac{j\omega L}{j\omega L + R} \\
&= \frac{-\omega L}{-\omega L + jR} \\
&= \frac{\omega L}{\omega L - jR} \\
&= \frac{\omega L}{\sqrt{(\omega L)^2 + R^2}} e^{j\tan^{-1}(R/\omega L)} \\
&= \frac{1}{\sqrt{1 + \left(\frac{R}{\omega L}\right)^2}} e^{j\tan^{-1}(R/\omega L)}
\end{aligned}$$