Lab 1

Introduction to Python, Numpy, Matplotlib, and Jupyter Notebooks

PHYS 221, Fall 2019

Python is a computer language that has a *very* wide range of applications, but in PHYS 221 you will use Python, along with the Numpy, Scipy, and Matplotlib modules, for a very limited set of tasks: the manipulation and plotting of data that we acquire in lab.

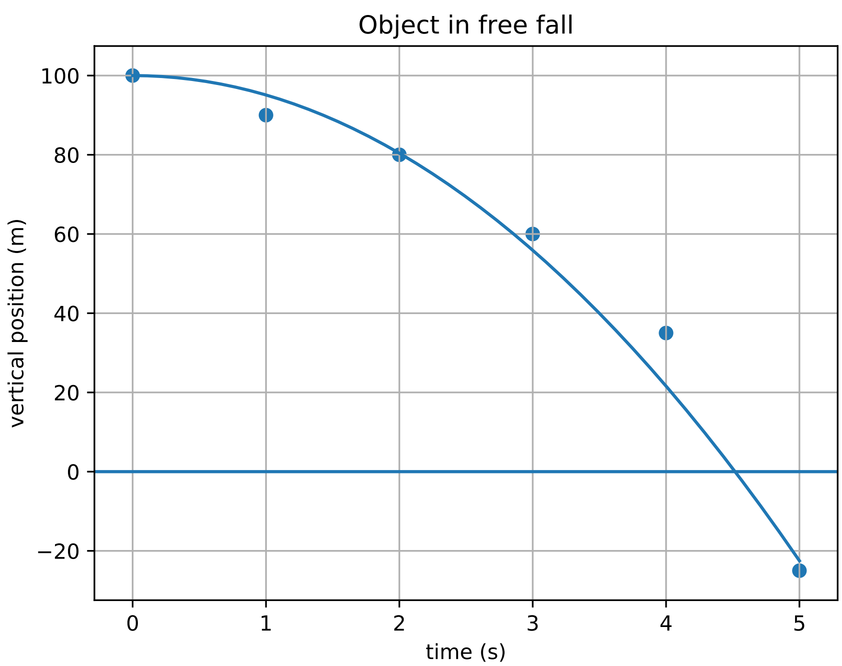
For today's lab let's consider a simple hypothetical experiment in which you measure the position as a function of time of a ball released from rest and falling under the influence of gravity. Let's assume that air resistance is negligible. Say that you obtained the following data:

t (s)	y (m)
0	100
1	90
2	80
3	60
4	35
5	-25

By the end of today's exercise you will be able to use Python tools to create a scatter plot of this data, together with a connected line showing the theoretical prediction. Along the way you will:

- create and manipulate Scipy arrays of data,
- define functions that take arrays as inputs,
- use "loops" to perform repeated calculations,
- plot data and functions,
- label axes, and
- make formatted comments within your Python programs.

The graph you make and hand in should look something like the following:



Getting Started

We will edit and run programs in the Python language from an environment called a Jupyter notebook. (There are many other ways to run Python programs, also know as *scripts*.)

- – For Bucknell Windows computers:
    `Windows Menu` → `Anaconda 3` → `Jupyter Notebook`

- For Linux computers:
  Start from a terminal and type the following on a command line

  ```
  jupyter notebook
  ```

  or something like

  ```
  jupyter notebook --browser=firefox
  ```

  if you want to specify a browser.

- For Bucknell Macs:
  Anaconda should appear as an application, and you should be able to start a Jupyter notebook running Python 3.

- A browser will appear running Jupyter. Click the button on the upper right labeled `New`. From the pull-down menu, select `Python 3`.

- You now have a Jupyter notebook running in which you can start writing a new program. Click on the text `Untitled` near the top, and give your notebook a name (that includes your name) — something like

  ```
  ligare_lab_1
  ```

- Jupyter notebooks are organized into "cells"; the first cell is the box outlined in green and contains the text `In[ ]`; more cells will appear as we go.

  In our work, the cells will be of two kinds: either

  - `Code` cells, in which you enter programming commands, or
  - `Markdown` cells, in which you can add formatted comments. (Shorter comments can be added in `Code` cells using the `#` sign; everything to the right of the `#` will be treated as a comment.)

  The kind of cell is indicated in the Menu Bar at the top. Change the initial cell to a `Markdown` cell, and enter the following in the cell

  ```
  ## My first notebook
  ```

  Then hit `Shift+Enter` (simultaneously). You should see something like a nice title, or section heading.

- Double click on the first cell, and change the number of `#` signs, and execute the cell by hitting `Shift+Enter` again. This is the first example of some of the formatting capabilities of the `Markdown` language.

- A new empty cell should have been automatically created below the first cell, and this should be a `Code` cell. In this cell, type

```
print(6 + 5)
```

and execute the cell. The output below the cell should make sense.

- Another `Code` cell should appear. In this cell type

  ```
  print(cos(0))
  ```

  and execute the cell.

  You get an error message because Python is a pretty bare-bones language, and doesn't, on its own, know about math functions like cosine and sine. But there many, many, *modules* that can be imported into a Python program for a wealth of applications. You will need to import the numerical python module called `numpy` for every lab we do.

- Single click on your title cell, and select

  `Insert` → `Insert Cell Below`

  In the new `Code` cell that appears, type

  ```
  import numpy as np
  ```

  and execute the cell.

- Return to the cell containing the `cos(0)` statement and modify it slightly so that it reads

  ```
  print(np.cos(0))
  ```

  and re-execute the cell. You have told python to use the `numpy` module to calculate cosines (and many other mathematical functions).

Python as a Calculator

Type the following commands, or sets of commands into cells, and execute the cells.

- `print(2*3)`
- `print(2**3)`
- `print(x-y)`
- ```
  x = 3
  y = 5
  print(x-y)
  ```
  ```

- `print("Hello World!")`
- `print("x+y")`
- `print( (y+x)*(y-x) )`
- `print(np.sin(pi/2))`
- `print(np.sin(np.pi/2))`
- `print(15/4)`
- `print(15//4)`
  (The `//` operator is called *integer division*. Can you see what is happening here?)

Assignment Operator

The 'equals sign' in Python (and many other programming languages) is not the same as in mathematics. The math equation $x = x + 6$ has no solution. Now try this:

```
x = 1
x = x + 6
print(x)
```

The 'equals sign' is an *assignment operator* — it *assigns* a new value to the variable `x`.

`For` Loops

Loops are how we tell the computer to do a particular task (or a slight variation of it) over and over.

```
for i in range(10):
    print(i)
    j = i**2
    print(i, j)
```

The indentation is essential. Jupyter will automatically indent the first line after a conditional statement, but you must "unindent" when you get to the end of the statements that are to be repeated.

Numpy arrays

Python has several different species of ordered sequences of items. In PHYS 221 we will focus on Numpy arrays, a data type that is available after importing the `numpy` module. (The same kind of arrays are available after importing the `scipy`.) For more about Numpy/Scipy arrays, see `http://www.scipy-lectures.org/intro/numpy/array_object.html`.

Making arrays 'by hand'

Enter the following arrays in cell, and execute the cell:

```
a = np.array([0,1,2,3])
b = np.array([[0,1], [1,4], [2,7], [3,10], [4,1]])
c = np.array([4,5,6,7])
```

Then enter and execute the following commands, one at a time, and see if you can figure out how things are working.

- `a`
- `print(a)`
- `a**2`
- `np.sqrt(a)`
- `np.sum(a)`
- `a*c`
- `c[0], c[1]`
- `c[2:]`
- `b[1]`
- `b[2][1]`

Some 'ready-made' arrays

Enter and execute the following commands, one at a time, and see if you can figure out how things are working. The *Scipy Array Tip Sheet* can be very useful when you're trying to make some standard arrays (`http://pages.physics.cornell.edu/~myers/teaching/ComputationalMethods/python/arrays.html`). Note that the authors of this page have imported `numpy` arrays via the *scipy* module, but they are really `numpy` arrays. (This method works, but is deprecated.) In using examples from the *Tip Sheet* simply replace instances of `sp` with `np`.)

- ```
  c = np.ones(5)
  print(c)
  ```
- ```
  d = np.zeros(5)
  print(d)
  ```
- ```
  e = np.linspace(0,1,11)
  print(e)
  ```

Fill an existing array

```
for i in range(len(d)):
    d[i] = i**3

print(d)
```

Create and fill an array in one step

```
g = np.array([i**3 for i in range(5)])
print(g)
```

Defining Python functions

A Python function is a lot like a math function — it takes one or more inputs, called arguments, does something with them, and typically *returns* an output. In your Jupyter notebook define a function as follows:

```
def my_function(r):
    # lots of calculations could go here
    return 6*r

my_function(2)
```

Look at the output. Now try

```
x = 7
r = 4
```
```

```
my_function(x)
```

Is the result what you expected?

There's more than meets the eye with this function. We didn't tell the function what type of variable `r` was. Let's make an array:

```
v = np.array([1,2,0])
my_function(v)
```

Now look at the output. The function `my_function` will return a number when the argument `r` is a number, but it returns an array when the argument is an array.

Graphs

Importing Matplotlib

Go back to the cell near the top of your notebook where you imported `numpy`. Add the following lines to the cell and execute it again.

```
import matplotlib.pyplot as plt
%matplotlib notebook
```

The first line imports the plotting module. When imported this way, every command from the `pyplot` module must be preceded by `plt` as a prefix.

The second line is not actually python code. It's a *magic command* that tells Jupyter to display the plots in the notebook, rather than popping up an external window.

Plotting a function

If we want to have 200 points in our plot in the range of `x` values from zero to some adjustable variable value named `l`, we can do that with

```
l = 10.0  # or whatever
x = np.linspace(0, l, 201)
```

But don't print out this array! It's too big. Instead, we'll use it to plot the function $y = x^2$. We can just add the line

```
y = x**2
```

to create the array of y values. Then we create a simple plot (with the points connected) as follows:

```
plt.figure()
plt.plot(x,y)
```

We can add lots of stuff to our graph:

```
plt.figure()
plt.plot(x,y)
plt.xlabel('x')
plt.ylabel('y')
plt.title('Title')
plt.axhline(0)
```

If you want to plot unconnected data points, use `plt.scatter` instead of `plt.plot`:

```
plt.figure()
x = np.linspace(0, 1, 5)
y = x**2
plt.scatter(x,y)
```

Getting Help

Help on learning how to do things using Python, Numpy, SciPy, and MatplotLib is available, of course, on-line. The most authoritative information can be found at

- `https://www.python.org/`
- `https://www.numpy.org/`
- `https://www.scipy.org/`
- `https://matplotlib.org/`

The pages at these sites can be a little dense, but they have common organization, often with simple examples at the bottom. There are many other sites that provide tutorials.

If you know the name of a function, but you can't remember the exact syntax or all the options, you can simply pre-pend a question mark to the name of the function. Try executing a cell containing the following:

```
?np.linspace
```

Assignment

1. Start a new notebook and create a graph for the hypothetical experiment discussed in the introduction. Your notebook should have a introductory cell with your name and the date, and a brief discussion of what you accomplish in the notebook. You should get a graph that looks something like the one displayed on page 2 of this handout.
2. **Extra.** Add some more features to your graph. You could figure out to add error bars (say $\pm 10\,\mathrm{m}$) for all y-values; you could figure out how to add horizontal and vertical axes; you could figure out how to change colors of parts of your graph, ...
3. **Extra.** Figure out how to use Scipy tools to do numerical integration. As a test, integrate the function $f(x) = x^3$ from $x = 1$ to $x = 2$. Evaluate the integral by hand to check your work.
4. **Extra.** Figure out how to use Scipy tools to solve a system of linear equations. Find the solution to the equations

$$\begin{aligned} x + y &= 4 \\ 2x - y &= 7. \end{aligned}$$

5. **Extra.** Figure out how to use Scipy tools to solve ordinary differential equations. Integrate Newton's second law for a 0.5 kg mass on a spring with a spring constant of 2 N/m that is displaced from equilibrium by 1 cm and released from rest.

Hand-in instructions

1. You should hand in a notebook with your final graph and any "extra's" that you had time to complete. (You do not need to include the initial examples.)
2. Give the file a name that includes your name and the lab number; something like

   `ligare_lab_1.ipynb`

   (When you save a notebook from within Jupyter, it will automatically add the `ipynb` extension.)
3. Save a copy of the notebook in your Netspace (or a `git` repository).
4. Put a copy of the notebook in my Netspace drop box.

Notes to Instructors

Python was first used in PHYS 221 in Fall of 2018. Michele Thornley was the lecturer, and Marty Ligare led both lab sections.

There was a *very* wide range of computer experience and expertise in the class, from strong (and weak) C.S. majors, to Mech E's with experience in MatLab, to coding novices. M.L.'s strategy was to set a minimum expectation (re-creating the graph on p. 2), but providing lots of **Extra**'s for the more experienced students. (M.L. never used the word *minimum* with the students.) The minimum includes things that will be used multiple times in the course; the **Extra**'s were in some sense enrichment, and included things that won't come up in 221 lab. A perfect grade could be achieved for a well-documented version of the minimum.

M.L.'s expectation was that experienced coders would willingly go on to complete some of the **Extra**'s, and this was the case. No students raced through the minimum and tried to leave.

The **Extra**'s are intentionally short on instructions — figuring out to do things from available on-line documentation is an important skill. (M.L. was, of course, ready to help students sort through the on-line resources.) (The final **Extra**, using `odeint` to integrate a differential equation, is much harder; it's probably not a good fit for students who don't have a good understanding of differential equations and some previous Python experience.)

The only imports in the entire course will be from Numpy, MatplotLib, and Scipy. As of 2019, recommendations are to import Numpy for all arrays and basic math, and then add Scipy sub-modules as necessary (see `https://docs.scipy.org/doc/scipy/reference/api.html`). For example:

```
import numpy as np

from scipy import optimize
```

and then call functions from sub-module as

```
result = optimize.curve_fit(...)
```

Lab 2

Analysis of motion using numerical integration I

PHYS 221, Fall 2019

Introduction

Consider the motion of a single particle of constant mass m in one dimension, say the vertical (y) dimension. The familiar form of Newton's second law is

$$F_{\text{net}} = ma, \tag{2.1}$$

but Newton's second law is really a *differential equation* that must be solved to find the functions like $y(t)$, $v(t)$, and $v(y)$ that describe the position and velocity of the particle. To emphasize the character of Newton's second law, perhaps it's better to rewrite Eq. (2.1) explicitly as a differential equation. There are several forms that this equation can take. It can be written as the second-order differential equation

$$\frac{d^2y}{dt^2} = \frac{F_{\text{net}}}{m}, \tag{2.2}$$

or equivalently it can be written as the two coupled first-order equations

$$\frac{dv}{dt} = \frac{F_{\text{net}}}{m} \tag{2.3}$$

$$\frac{dy}{dt} = v. \tag{2.4}$$

The solutions of differential equations are functions, not numbers. In this lab you will start with an expression for the net force F_{net}, and you will solve the coupled differential equations in the form given in Eqs. (2.3) & (2.4) to get the functions $y(t)$ and $v(t)$.

In simple cases, the net force F_{net} might be constant, but in general it might be a function of the particle's position or its velocity, and it might depend explicitly on time, *i.e.*, $F_{\text{net}} = F_{\text{net}}(y, v, t)$.

Solving First-Order Differential Equations Numerically: Euler's Method

Sometimes you can solve the differential equations describing the motion analytically (as you will do in Exercises 2 and 3 below); this means that you will end up with *formulas* for $y(t)$ and $v(t)$. In other cases it will be difficult or impossible to find analytic functional forms for $y(t)$ or $v(t)$. In such cases you can get approximate solutions *numerically*. You are going to generate numerical solutions to differential equations step-by-step.

Starting from the definition of the derivative,

$$\frac{dv}{dt} = \lim_{\Delta t \to 0} \frac{v(t + \Delta t) - v(t)}{\Delta t}, \tag{2.5}$$

it is straightforward to show that for small time steps Δt

$$v(t + \Delta t) \simeq v(t) + \frac{F_{\text{net}}}{m} \Delta t. \tag{2.6}$$

This says that if we know the velocity (v) and acceleration ($a = F_{\text{net}}/m$) at some time t, we can calculate an approximate expression for the velocity a little while later at time $t + \Delta t$.

In a similar manner, it is straightforward to show that

$$y(t + \Delta t) \simeq y(t) + v(t)\, \Delta t, \tag{2.7}$$

which says that we can calculate an approximate expression for the position at time $t + \Delta t$ if we know that position and velocity a little while earlier at time t.

Exercises

In the following exercises you will calculate the motion for two simple physical situations. For the simplest example you will solve the problem three ways:

- you will derive exact analytic functions for $y(t)$ and $v(t)$,
- you will find approximate solutions for $y(t)$ and $v(t)$ at a number of discrete time points using a simple program that you write in Python, and,
- if there is time you will check your solution with an approximation calculated with more sophisticated methods.

In more complex situations you may not be able to find analytic solutions to the equation of motion (Newton's second law), and you will have to rely on approximate numerical solutions.

Here's the physical situation: Consider a ball dropped from the top of a 50 m tower. To begin, let's assume that the ball is dropped at time $t = 0$, with velocity $v = 0$, and let's ignore air resistance and any other complicating factors that you can think of.

1. Draw a free-body diagram for the ball in its flight, and determine an expression for F_{net}. (This task is trivial in this example, but the procedure is one that we will use many times, so let's get in the habit of doing it.)
2. Find a function $v(t)$ that is a solution of Eq. (2.3). Use the initial condition for the velocity at $t = 0$ to determine any constants in your function.
3. Find a function $y(t)$ that is a solution of Eq. (2.4) when you use the function $v(t)$ you just determined for the right-hand side of Eq. (2.4). Use the initial condition for the position at $t = 0$ to determine any constants in your function.
4. Convince yourself of the validity of the approximations in Eqs. (2.6) & (2.7) for small time intervals Δt, *i.e.,* show

$$v(t + \Delta t) \simeq v(t) + \frac{F_{\text{net}}}{m}\,\Delta t \qquad (2.8)$$

$$y(t + \Delta t) \simeq y(t) + v(t)\,\Delta t. \qquad (2.9)$$

5. Use the approximations you derived in the previous problem to fill in the blanks in the following table. Your entries should be correct to at least 3 decimal places.

t	$y(t)$	$v(t)$	F_{net}/m
0.0			
0.1			
0.2			
0.3			

6. Write a simple Python program in a Jupyter notebook that uses the approximations you derived in the previous problem to find approximate values of $v(t)$ and $y(t)$ at 11 times from $t = 0$ to $t = 1\,\text{s}$ that are separated by a time $\Delta t = 0.1\,\text{s}$. Here are some suggestions:

   - Include a cell at the top in which you import necessary modules.
   - Include a cell in which all physical constants are given symbols and values. Include things like the values of the the initial time, the final time, the number of time points at which you want to calculate values, the initial position, the initial velocity, and the gravitational field strength..
   - Include a cell near the top in which you define your functions; it's a good idea to have a function that returns the force.
   - Include a cell in which Numpy arrays are set up for the variables v, y, and t.
   - The repeated calculations can be carried out in a Python `for` loop. In each pass, your program should calculate a new element in the velocity array, a new element in the position array, and a new element in the time array.

7. Check that your Python results agree with the results you calculate "by hand."

8. Make a graph that includes your approximate solution for $v(t)$ along with your analytic solution for $v(t)$. This will involve defining a function for $v(t)$.

9. Make a graph that combines the graphs for your approximate solution and your analytic solution for $y(t)$ in a single plot.

10. Do the points in you numrical approximations lie on your theoretically detrmined functions? Investigate the quality of your approximation as you change the number of points you use in the time interval between 0 and 1 s.

Now you will repeat the process, but this time you will incorporate air resistance into the problem. Air resistance contributes a force that depends on the velocity of the particle; a simple approximation for this force that works well for one-dimensional motion at low speeds for some particles in some media is

$$F_{\text{drag}} = -Av, \tag{2.10}$$

where A is a constant. We will use this expression for F_{drag} not because it is realistic for all circumstances, but because it is an easy velocity-dependent drag force to work with, and any program that works for this force should be easy to generalize for a more complicated velocity-dependent force.

You will use exactly the same strategy you used in the previous exercise. The only difference is that the force on the ball will not be constant. This should require only a few changes to the notebook you wrote for the previous exercise.

11. Draw a free-body diagram for the ball in its flight, and determine an expression for the net force F_{net}.

12. Determine an expression the terminal velocity of the ball.

13. Use the approximations you derived in Exercise 4 to fill in the blanks in the following table. (Use $m = 1.0\,\text{kg}$, and $A = 0.5\,\text{N}\cdot\text{s}\cdot\text{m}^{-1}$. This value for A is not a value based on reality — it's a value based on getting nice numerical results in this exercise.)

t	$y(t)$	$v(t)$	$a(t) = F_{\text{net}}(t)/m$
0.0			
0.1			
0.2			

14. Write a simple Python program in a Jupyter notebook that uses the approximations you derived in Exercise 4 to find approximate values of $v(t)$ and $y(t)$ at 11 times from $t = 0$ to $t = 1s$ that are separated by a time interval of $\Delta t = 0.1s$. (This should be a simple modification of the program you wrote for the case of free-fall with no air resistance.)

15. Make graphs of your approximate solutions for $y(t)$ and $v(t)$.

16. Combine the graph for your approximate solution for $v(t)$ with air resistance and your analytic solution for $v(t)$ with no air resistance.

17. Combine the graph for your approximate solution for $y(t)$ with air resistance and your analytic solution for $y(t)$ with no air resistance.

18. Investigate the quality of your approximation as you change the number of points you use in the time interval between 0 and $1s$. Do you get the expected terminal velocity?

We are taking no measurements today, but you'll be able to use your code in concert with observations next week that will enable us to see how well we have modeled a situation much like this one.

What to include in your notebook

We expect you to complete the first few steps (the "pre-Python" steps) in each of the two scenarios by hand in your physical notebook, and indeed, these steps will be a useful reference as you check the results from your Python code for the same scenarios. The Jupyter notebook that you hand in should include your final graphs for $y(t)$ and $v(t)$ for the cases of no air resistance and with air resistance, and all other cells needed to generate the graphs. Preliminary work can be cut from your notebook. Don't forget to add comments to your Jupyter notebook.

The last cell of your Jupyter notebook should be a short abstract of a few sentences. This is to be a short summary of the system you investigated, the method you used, did and the results you obtained.

Extras

- Modify your code to integrate Newton's second law for a 0.5 kg mass on a spring with a spring constant of 2 N/m that is displaced from equilibrium by 1 cm and released from rest. Further modify your code to study the same mass-spring system, but this time add a velocity dependent drag force, $F_{\text{drag}} = -bv$, where b is a constant. (You can start with $b = 1\,\text{kg/s}$, but try other values too.)
- Use Wolfram Alpha to solve your equations of motion.
- Use the Scipy function `odeint` to solve your equations of motion.

Notes to Instructors

The whole idea of programming is new to most of the students in 2008. For example, the translation of Euler's equations to statements like `y[[i]] = y[[i-1]] + v[[i-1]]*dt` was not easy for everybody.

Mathematica lists are indexed 1..N and not 0..(N-1), so code that attempts to call position[[0]] will lead to errors without a clear indication of why.

Not all students "get" the idea that they should check the computer output against the data in the table they calculate by hand.

It is worth emphasizing clearly that they are comparing two things: a list using Euler's method which is their simulation and a continuous function that is the analytical model. For the Euler's method, I mention that they will need to define their lists with Table before using For to determine their values.

Lab 3

Analysis of motion II: Comparison of numerical approximation with experimental measurements

PHYS 221, Fall 2019

Introduction

In this two-week lab experiment we will analyze the motion of real balls falling under the influence of gravity, and you will include the effect of air resistance. There are two parts of this lab: the collection of data, and the analysis of this data using Python in Jupyter notebooks that will be very similar to those you developed in the previous lab exercise. In the experimental section, you will drop balls from the balconies overlooking the foyer between the Biology and Chemistry buildings, and measure the time it takes for the balls to reach the floor. In the analysis section, you will build on your work from the previous exercise using a realistic expression for the drag force, and calculate the position of the ball as a function of time, $y(t)$. The two sections of this exercise may be done in either order.

Experiment

In this lab you investigate how long it takes for a small ball to fall to the ground from a height of several meters.

1. Go to the foyer between Biology and Chemistry and drop balls from the two

balconies. Your goal is to obtain the best possible measurement of the time it takes for a variety of balls to hit the ground. You will be provided with measuring tapes and stop watches. Try various methods and decide upon the best technique.

Be sure to practice and repeat your measurements enough times so that *i*) you become proficient at making consistent measurements, and *ii*) you get a feel for the range of values that are reasonable given the equipment you have. Determine the uncertainty due to random errors in your timing measurements. (Each member of a group should try to make some timing measurements.) You should get "good" data on several balls.

2. Determine your group's best value for the time for each ball and each height, **including a justified estimate of your statistical uncertainty**. You should strive for a statistical precision of better than 5%.
3. Comment in your notebook on any *systematic* sources of error that are not accounted in the statistical analysis of your data.
4. Measure any physical properties of the ball that you will need to compare your data with your theoretical prediction.

Analysis

In this part of this lab you will continue your analysis of the motion of a single particle falling under the influence of gravity, and you will include a realistic expression for air resistance. You will build on your work from Lab #2 and use your program to calculate something you can actually measure. The goal is to determine theoretically the position of the ball as a function of time, $y(t)$. This will allow you to predict the time it takes for a ball to fall to the floor from each of the levels of balconies in the foyer between the Biology and Chemistry buildings, and see if your predicted times are consistent with measured times.

Air resistance contributes a force that depends on the velocity of the particle. For spherical objects the size of ping pong ball falling in air at the speeds encountered in this lab, the *magnitude* of the drag force is proportional square of the speed:

$$F_{\text{drag}} = \frac{1}{2}\rho C_d A v^2, \tag{3.1}$$

where A is the cross-sectional area of the ball, v is the speed of the ball, ρ is the density of the air (which depends on the temperature and pressure, but which is approximately $1.18\,\text{kg/m}$), and C_d is a shape-dependent drag coefficient ($C_d \simeq 0.47$

for a sphere). You will have to make slight modifications to the program you wrote in Lab #2 to incorporate this form of a velocity-dependent force.

As before, the differential equations we are trying to solve are

$$\frac{dv}{dt} = \frac{F_{\rm net}}{m} \tag{3.2}$$

and

$$v = \frac{dy}{dt}, \tag{3.3}$$

and we will be using the approximations (that are valid for small time steps Δt)

$$\begin{aligned} v(t+\Delta t) &\simeq v(t) + \frac{F}{m}\Delta t, && (3.4)\\ y(t+\Delta t) &\simeq y(t) + v(t)\Delta t. && (3.5) \end{aligned}$$

Note that for our specific case, $F_{\rm net} = F_{\rm net}(v)$, i.e., the net force depends on the ball's speed.

Analysis Exercises

1. Draw a force diagram for the ball in its flight, and determine an expression for $F_{\rm net}/m = a(t,x,v)$ that you can use in the right side of Eq. (3.2).
2. Use your results from the previous problem to determine the *terminal velocity* of the ball.
3. For numerical convenience, use the values $F_{\rm drag} = 0.1v^2$, $m = 1$, and $y(0) = 20$ to fill in the entries in the following table:

t	$y(t)$	$v(t)$	$a(t) = F_{\rm net}(t,y,v)/m$
0.0 s			
0.1 s			
0.2 s			
0.3 s			

4. Write a simple Python program to find approximate values of $v(t)$ and $y(t)$ and $F(t)/m = a(t,x,v)$ at 11 times from $t = 0$ to $t = 1$ that are separated by a time $\Delta t = 0.1$ s. You should base this program on your successful program from Lab #2. Be sure to check the values calculated by your program against your entries in the table above. Here are some suggestions for your program:

- Include a cell in which all physical constants are given symbols and values. Include things like the value of the initial time, the final time, the number of time points at which you want to calculate values, the initial position, the initial velocity, and the value of the drag constant C_d, etc.
- Define a Python function that calculates $a = F_{\text{net}}/m$ as a function of velocity.
- Include a cell in which Scipy arrays are set up for the variables v, y, and t.
- The calculations can be carried out in a Python `for` loop. In each pass your program should calculate a new element in the velocity array and a new element in the position array.

5. When your program is working and giving the same answers as your entries in the table, change the parameters to reflect the physical situation you are studying, *i.e.*, one of the real balls in free fall. (If you haven't measured the properties of the balls yet, do so now.) Make a graph of your approximate solution for $y(t)$ and $v(t)$.
6. If the results of your program look reasonable, increase the number of points you calculate between $t = 0$ and $t = 1$ (or equivalently decrease your time step Δt) until your solution doesn't change with smaller values of Δt.
7. Use the following checks to make sure your program is giving you reasonable results:
   (a) Set the constant $C_d = 0$ in your expression for the drag force and make a graph that shows the solution from your program **and** the analytic solution you derived in Lab #2 for the case of no air resistance. Comment on the results of this exercise.
   (b) Make a graph that shows your numerical solution for $C_d = 1$ **and** the analytic solution you derived in Lab #2 for the case of no air resistance. Does your solution deviate from the analytic solution in a "reasonable" way? (You may have to adjust some scales to combine graphs successfully. If you don't know how to do this, get some help.)
   (c) Does your solution reach a terminal velocity? Make whatever adjustments are necessary so that you can check whether your program gives you the terminal velocity you expect from the calculation you did in Exercise 2 above.
8. Adjust the parameters in your Jupyter notebook so that your results will correspond to the physical situation that you have measured. For each of the

balls you drooped create plots of

(a) the position $y(t)$ in the case of no air resistance,

(b) your prediction of $y(t)$ with air resistance,

(c) the velocity $v(t)$ in the case of no air resistance, and

(d) your prediction for for $v(t)$ with air resistance.

Combining experiment and theory

1. Examine your theoretical data to get an estimate of the time it takes each of the balls to fall to the ground. For each case, compare the theoretical prediction to the time that you measured experimentally. Are the predictions and the experimental values consistent within uncertainty? If not, comment about what you think might have contributed to the difference, either from the prediction or the measurements. Address this in your notebook.

2. Find the experimental difference in the fall times for a "light" and a "heavy" ball.

   Compare your theoretical and experimental values for the *difference* in the fall times of a "light" and "heavy" ball. Are they consistent? If not, are there things that you have left out of your theoretical model that might have affected your calculated times? If you have time, test the effect of things you have left out.

   Question: Why might a comparison of the difference between the times be better than a comparison of the actual times themselves?

3. Compare your time differences with those found by your classmates. If it is appropriate, combine your results to get a class result for the time difference (with an uncertainty).

4. Compare your theoretical and experimental values for the *difference* in the fall times of a "light" and "heavy" ball. Are they consistent? If not, are there things that you have left out of your theoretical model that might have affected your calculated times? Test the effect of things you have left out.

5. Galileo is purported to have dropped balls from the leaning tower of Pisa to show that all balls fall at the same rate, independent of mass. Do your results agree with those attributed to Galileo? Do you think his masses hit the ground at the same time? Was he a liar? Was he wrong? Can you justify his conclusions in some limit?

Optional extra exercise

If you have time, extend your program so that it can handle projectile motion in two dimensions. For uniformity, let's all start with the case of $v_0 = 10$ m/s directed at an angle of $\theta = 30°$ above the horizontal.

What to include in your notebooks

You will be turning in two notebooks: a physical lab notebook and a Jupyter notebook. exercises in the previous section. Your notebooks should include:

- answers to all the numbered exercises in the handout,
- a description of what you actually did, with commentary on technique, measurements, and uncertainties (systematic and statistical),
- a quantitative summary of your experimental results,
- an assessment of you well the results of your numerical integration model matched your experimental observations, and
- an abstract summarizing the experiment.

Notes for Instructors

One of Galileo's "accounts" is in *Dialogues Concerning Two New Sciences* (p. 62 in my copy; sections 106-107).

August 9, 2020

Lab 4

Functional Relationships from Simple Oscillator Data

PHYS 221, Fall 2019

Introduction

Suppose you have data on two related quantities, and you want to see if you can determine a functional form for the relationship. One way is to "look for" specific kinds of relationships. For example, if you suspect that two variables have a linear relationship you can simply plot them to see if you are right.

If you suspect that there might be a power-law relationship, *i.e.*,

$$g(v) = av^b, \tag{4.1}$$

where a and b are constants, it will be a little more difficult to see if you are right by simply plotting g vs. v, and it wouldn't be easy to get a value for a or b from the graph. But there are some straightforward techniques you can use to make things easier. For instance, you can take the logarithm of both sides of Eq. (4.1) to get the relationship

$$\log(g) = \log(a) + b\log(v). \tag{4.2}$$

If you plot $\log(g)$ vs. $\log(v)$, rather than g vs. v you *will* get a linear relationship *if* your hunch about the functional form of the relationship is correct.

If you suspect that the relationship has the form

$$g(v) = a\exp(-bv) = ae^{-bv}, \tag{4.3}$$

where once again a and b are constants, you can play the same sort of game. You will have to determine for yourself the details of a method to test whether some data fits this form.

Python/Numpy/SciPy notes

You have performed least-squares fits to find best-fit lines in PHYS 211 using Excel. One way to do this (and more) using Python tools is to ‘ use a curve fitter, appropriately named `curve_fit`, that is part of the `scipy.optimize` sub-module.

- In the cell in which you import modules include the line

```
from scipy import optimize
```

- Define a function for a straight line, something like

```
def f(x, intercept, slope):
    return intercept + slope*x
```

- Assuming that you have an array `x` containing the data for your independent variable (horizontal), and an array `y` containing the data for your dependent variable (vertical), and potentially an array `u` for your uncertainties), you can fit the data to your function `f` using the command

```
popt, pcov = optimize.curve_fit(f, x, y)
```

  or, if you want to account for the uncertainties,

```
popt, pcov = optimize.curve_fit(f, x, y, sigma=u)
```

  The `curve_fit` functions returns two arrays. The optimum fit parameters (the best values of the intercept and slope) will be returned in the array `popt`, and information about the uncertainties in the fit parameters will be returned as the diagonal elements in the 2×2 array `pcov`.

- If you want to create a plot of data with error bars, use

```
plt.errorbar(x, y, yerr=u, fmt='o')
```

- Remember that there lots of Numpy tools for manipulating arrays. You could record each data point as a triple of numbers, corresponding to `[x1, y1, u1]`, where `u1` is the uncertainty, and your data might look like

```
data = np.array([[10, 4, 0.5], [20, 8, 0.5], [30, 11.5, 0.5]])
```

  To get an array with just the x-values, take the ‘0’ element of the transpose of the array `data`:

```
x = np.transpose(data)[0]
```

and similarly

```
y = np.transpose(data)[1]
u = np.transpose(data)[2]
```

Experiments

There are three parts to this lab. In each part, focus on the most useful way to plot the data to determine the quantity of interest. You should discuss each set of plots and results with your instructor.

Determination of a Spring Constant

Remember that the magnitude of the force that a linear spring exerts on an object depends on the the extension of the spring x from its equilibrium length,

$$F_{\text{spring}} = k|x|. \tag{4.4}$$

- Discuss with an instructor your strategy for determining the spring constant.
- Take data, and enter your it into a Numpy array (or arrays) in a Jupyter notebook.
- Implement your plan for determining the spring constant.
- Do most of your data points fall within an error bar of your best-fit line?
- Give a properly formatted result for your spring constant.

Dependence of frequency of oscillation on mass

- Measure the period of oscillation of your spring when a variety of masses is hung from the spring. Use small masses for this experiment — say, from 10 g up to about 60 g.
- Make a graph of frequency vs. mass. Do you get a linear relationship?
- Make a graph that helps you determine whether your data is described by a power law relationship like that in Eq. (4.1).

- Determine a functional form that describes the relationship between frequency and mass. Use the fitting function in Eq. (4.2) with your data to determine the value of the exponent in power law.
- Does the exponent make sense?

Functional form of the decay of an oscillator

- As a class we will measure the amplitude as a function of time for a slowly decaying oscillation. Record the data.
- Plot your data. Is the relationship between amplitude and time linear? Is it described by a power law?
- Take the logarithm of both sides of Eq. (4.3) to determine a linear relationship that you can check by plotting your data.
- What is the damping constant b for the air-track oscillator?

Notes to Instructors

General

In 1999 Sally and Marty did this for the first time. Oscillations hadn't been covered in class, so this was purely a "discover the functional relationship" lab. Not one student knew what the functional form of period vs mass "should" be, and that made the lab great. (We didn't push them to dredge up their memories of PHYS 211, or equivalent, but we did inquire about it.) We left the interpretation of the intercepts in the log-log and semi-log plots to future labs. None of this was review to the students. In 2007, this was done at the same time as oscillations in class, and it works well with this sequence too.

Spring Constant

We used the air track apparatus to do this, but without using the air. We simply secured one of the carts to the track with a rubber band, attached a spring to the cart, attached a string to the other end of the spring, hung the string over one of the pulleys at the end of the track, and hung small weights on a hangar attached to the string. There isn't actually any need to use the air track: hanging the spring vertically from any fixed mount would work.

The spring constant data isn't really used in this lab after it is measured, but this data is used in the *next* lab. (It's a good idea to label the springs so data can be used later.) But it's also worthwhile to include fitting for the spring constant here for a couple of reasons:

1. it's a nice linear fit,
2. it's good to discuss "best" ways to determine k from a data set (many students want to find a whole bunch of k's from pairs of data points, and then average them), and
3. it's a nice introduction to the `curve_fit` function (or the *Mathematica* fitting functions).

The natural tendency is to plot F *vs.* x. M.L. talked to individual groups about errors, plotting, least-squares fitting, etc., but didn't make a big deal about it at this time.

Old PHYS 211 problem comes up: Is it better to time 50 periods once, or 10 periods 5 times?

Period *vs* mass

NOTE: You could also measure frequency *vs.* mass.

- Advantage: The graph is more obviously non-linear than graph of period *vs.* mass. This makes "conversion" to linear form more dramatic. (It's also easier to use with "bad" data.)
- Disadvantage: The graph is more obviously non-linear than graph of period *vs.* mass. If you want to make the point that a graph can appear at first to be pretty linear, but closer examination can reveal small systematic deviations from a straight line, then the period data is better to use.

It's extremely important to start with masses that are as small as possible. Otherwise the square root function looks too linear. You are looking at $m = m_0 + \Delta m$, and if $\Delta m/m \ll 1$, then

$$T = c\sqrt{m} \simeq c\sqrt{m_0}\left(1 + \frac{1}{2}\frac{\Delta m}{m_0}\right).$$

M.L. used masses in the range of "empty hangar" up to 50 g. Curvature is slight, but readily apparent with good data.

When doing the log-log plot of the period/mass data, the slope is sensitive to the small amount of mass in the spring (and the pulley if you're doing this on the air tracks). Just including the mass of the weights leads to a slope of less than 0.5. You can make things "better" by pulling out the formula on p. 61 of French's *Vibration and Waves* that includes mass, i.e.,

$$\omega^2 = \frac{k}{m + M/3}.$$

Amplitude *vs* time

For this we did use the air tracks with air. We set up a cart attached to a spring on both ends, and the springs were attached to fixed points. The oscillation decays over a long enough time that the students can easily note the position of the maximum extent of the oscillation at a given time. Data collected in this experiment really looks nonlinear if you record over several half-lives, and looks bad in a log-log plot, but looks very linear (especially for the first few half-lives) in the semi-log plot.

Lab 5

How Good is Your Fit? An Introduction to Residuals

PHYS 221, Fall 2019

Introduction

In many scientific experiments you have a mathematical model that expresses a relationship between physical quantities, and the goal of an experiment is to determine some parameters in the model. For example, in Lab 4, *Functional Relationships from Simple Oscillator Data*, we experimentally determined that there was a linear relationship between the extension of a spring and the force exerted on the spring:

$$|F_{\text{spring}}| = k|x|, \tag{5.1}$$

and we then determined a value for the spring constant.

In this lab you will take the same kind of data you did in Lab 4 and measure the restoring force for a different kind of oscillator, and then you will look more closely at the relationship between the linear extension and the restoring force.

Residuals

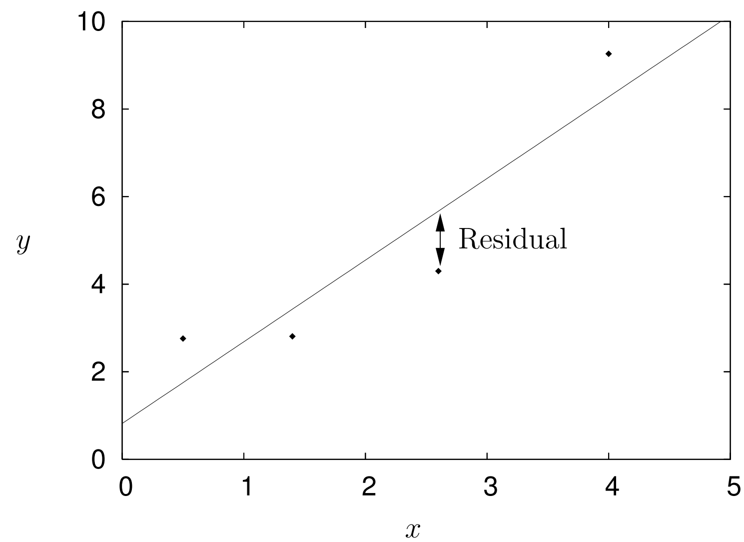
In assessing whether data actually fits an assumed model, it's useful to examine *residuals*. Residuals are the difference between the actual data and the values predicted by the model. Let's imagine that you recorded the following data for two experimental variables x_i and y_i:

i	x_i	y_i	$y(x_i)$	Residual
1	0.5	2.76	1.76	1.00
2	1.4	2.81		
3	2.6	4.30		
4	4.0	9.26		

The x_i's are assumed to have negligible uncertainty, so they are plotted along the horizontal axis. A least-squares fit of the data to a line gives the function

$$f(x) = 1.87x + 0.82,$$

which is plotted along with the data in the graph below.



The *residuals* are the vertical differences between the data and the line that is presumed to fit the data. A mathematical representation of the i^{th} residual r_i is

$$r_i = y_i - f(x_i).$$

A graph of the residuals *vs.* x should show a random scatter of points about zero, and the magnitudes of the residuals should be consistent with the estimated uncertainties in the measured values y_i.

Procedure

1. Fill in the empty boxes in the sample data table above.

2. Carefully measure the vertical position of the suspended mass for about 15 masses between 0 and 150 g.

3. Estimate the uncertainty in your measurements of the vertical positions.

4. In your data set the masses are known to high precision, and your measurements of position have some accompanying uncertainty. Therefore make a graph of position *vs.* mass (position on the vertical axis; mass on the horizontal axis).

5. Fit your data to a straight line, and plot the resulting line on the same graph with your data. Does the fit look good?

6. Make a graph of your residuals. What does this graph say about the quality of your fit to a linear function? This can be accomplished with a single-line plotting command like those that you have used before — if you need some help, just ask.

7. Think about the physical system you are investigating. Why does your measured relationship exhibit nonlineararity? Draw a torque diagram on the large disk, and use it to predict the functional relationship between position and mass for your system. Discuss your ideas with your instructor.

8. Fit your data to the functional form you predict, and plot your residuals. Because the relationship is nonlinear in some of the fit parameters, you will need to give `curve_fit` an initial estimate of you parameters:

   ```
   guess = np.array([p1, p2, ...])}
   popt, pcov = optimize.curve_vit(f, m, y, guess)
   ```

   where p1, p2, ... are your numerical estimates of the parameters.

9. What does your graph of the residuals say about your predicted relationship and your data?

Notes to Instructors

This was a new lab in 2006 thrown together by M.L. A couple students really liked it, and there was some evidence that it had some value in the 2007 version of PHYS 329.

This is a simple non-linear oscillator constructed from the big gyroscopes. The central bar of the gyroscope is clamped in a horizontal position, and things are arranged so the big disk hangs out over the edge of the table. A couple of standard disk masses totaling about 100 g are taped to the big disk very near the rim. A piece of string is draped over the top of the disk; one end is taped down, and the free end is draped over the top of the disk and allowed to hang vertically. Masses are hung from the string and displacements are measured. This is very much like the measurement of the spring constant in the *Functional Relationships* lab. (For best results, very fine string or thread should be used.) The data looks pretty linear on a gross scale, but residuals show that it deviates systematically from a straight line. In addition, the magnitude of the residuals is *much* greater than a reasonable estimate of the uncertainty in the position measurements. Another thing to talk about is correlation coefficients. The fits to a line have a very high value of R^2, and this presents a good opportunity to talk about why R^2 is *not* a good measure of goodness-of-fit to a linear model in this kind of situation.

October 2018: I've been puzzled over the years about why there is such a prominent quadratic contribution when we fit the data;

Lab 6

Rotational Oscillations I

PHYS 221, Fall 2019

Introduction

In the laboratory exercise *Functional Relationships* you investigated the dependence of the period of a simple harmonic oscillator on the mass of the oscillating object. For rotational oscillators the period depends not only on the magnitude of the mass, but also on the distribution of the mass relative to the center of rotation. You will use the same techniques you used in the *Functional Relationships* experiment to deduce the functional form of this dependence.

From the results of the experiment *Functional Relationships* (as well as theoretical derivations) you know that there is a power-law relationship between the frequency f of a simple harmonic oscillation and the mass m:

$$f = bm^{-1/2}, \tag{6.1}$$

or equivalently, a power-law relationship between the period T and mass m:

$$T = cm^{1/2} = c\sqrt{m}, \tag{6.2}$$

where c is a constant. (From the theory of simple harmonic oscillation, it's straightforward to figure out a formula for the constant, c.)

In this lab you will investigate the period of a rotational oscillation as a function of the position of the mass of an extended object. The masses will sit on a rotating plate, and the plate has its "own" period of oscillation. You should experiment with the 8 masses at your bench (totaling a little over 1000 g), placing them at various positions on the plate, and see how the position affects the period. From your qualitative experiments, and your results from the *Functional Relationships* experiment, it should be plausible that we can write the period of oscillation as

$$T = c_1\sqrt{I_0 + mr^\alpha}, \tag{6.3}$$

where I_0 is a property of the plate alone, and the exponent α is unknown. In this lab you will determine the constant α.

To facilitate your work, it is convenient to rewrite Eq. (6.3) as

$$\left(T^2 - c_1^2 I_0\right) = c_1^2 m r^\alpha. \tag{6.4}$$

Procedure

1. Place your eight masses as close to the center of the plate as possible (so that $r \simeq 0$) and measure the period of oscillation.
2. Distribute the same amount of total weight symmetrically around the plate at some known radius r and re-measure the period.
3. Repeat your measurement of the period for at least 6 different radii for the masses.
4. Analyze your data to see if it has the functional form of Eq. (6.4). If it does, determine the constant α.

Notes to Instructors

This experiment uses the trifilar pendulums that will be used later to measure moments of inertia. Marty began with a simple demonstration that the position of the mass on the oscillating plate makes a dramatic difference in the period. Then he tried to motivate Eq. (6.3) as a plausible functional form.

This lab involves no new techniques. All plotting commands have been used before; the log-log plotting idea was used previously (although in this lab we're taking log's of a manipulated quantity).

Results are pretty good. The error introduced in not being able to place all of the mass at $r = 0$ isn't terribly significant. If you stack a few heavy cylindrical weights on top of each other, the effective moment of inertia is on the order of $0.5 \times m \times (1\,\mathrm{cm})^2$, which is about 1% of the moment of inertia when the masses are distributed out at $r = 10\,\mathrm{cm}$.

We did this lab in conjunction with a wrap-up of Lab 3. See handout for Lab 3 wrap-up.

Motivation of Lab:
Sally began discussion of this lab at a time when the class was covering rotational motion. She began by reminding people of the results from the previous lab, the period of a pendulum varies according to the square root of the mass. It is then a natural extension to discuss the trifilar pendulum as the rotation of an extended object (the plate). As such, it is straightforward to motivate that, by complete analogy, it can expected that

$$T \propto \sqrt{I_0}$$

where I_0 is the moment of inertia of the plate (making up the pendulum).

The goal of the experiment is framed as: Given that the period of the pendulum goes as the square root of the moment of inertia, we can use this fact to probe the functional relationship of the moment of inertia. To do this, masses will be added to the plate of the trifilar pendulum; they will be added at a fixed radius. The goal is then to determine the power law of the added moment of inertia. Specifically, what is α if $I = mr^\alpha$?

This is done by beginning with the assumption

$$T \propto \sqrt{I_0 + I_{\mathrm{added}}}$$

where $I_{\mathrm{added}} = mr^\alpha$.

Then the task becomes, "What must be plotted so that α can be extracted from the data?" This leads naturally to the expression $(T^2 - c_2 I_0) = c_2 m r^\alpha$.

Aside: A value of I_0 must be determined which carries through all the calculations in this experiment. After some discussion on the board, it becomes apparent that I_0 can be determined by measuring the period when the radius $r \to 0$. This can be accomplished in two ways which appear equivalent to students at first: 1) take off the six masses (constituting the added masses to the plate) and then measuring the period, or 2) put all six masses as close to the center of the plate ($r = 0$) and measure the period. Are these methods equivalent? I had each group measure the period using both methods. It turns out that there is a difference of almost a factor of two. I then tell them to think about the 'system' and which case keeps the system as constant as possible with only a rearrangement of the masses. Clearly, the tension in the strings of the trifilar pendulum is different when the masses are removed. Keeping the masses on the plate and moving them to the center does not introduce any changes to the system.

Lab 7

Rotational Oscillations II

PHYS 221, Fall 2019

Introduction

In *Rotational Oscillations I* you investigated how the period of rotational oscillations of an extended object depended on the position of the mass. You measured the period of oscillation of a "trifilar pendulum" consisting of a metal disk suspended by three strings, with some "extra" mass m placed on the disk. You found that

$$T = c_1 \sqrt{I_0 + mr^2}, \tag{7.1}$$

where I_0 is a property of the plate, and r is the distance of the "extra" mass from the axis of rotation.

The *moment of inertia*, I, (also known as the *rotational inertia*) of an extended object like the plate is just the sum of lots of terms like the mr^2 in Eq. (7.1), with a term for every little piece of mass that comprises the object:

$$I \equiv \sum_i (r_\perp)_i^2 \, \Delta m_i, \tag{7.2}$$

where $(r_\perp)_i$ is the perpendicular distance from the i^{th} mass element to the axis of rotation. In the limit of a continuous mass distribution this becomes

$$I = \int r_\perp^2 \, dm. \tag{7.3}$$

In Kleppner and Kolenkow the variable $r_\perp$ is given the name ρ, but in this lab we will reserve the symbol ρ for volume mass density.

The equation of motion for rotations about a fixed axis is a simple generalization of Newton's second law. For a mass m on a linear spring with spring constant k we have

$$m\frac{d^2x}{dt^2} = -kx, \tag{7.4}$$

which has the solution

$$x(t) = A\cos\left(\sqrt{\frac{k}{m}}\,t + \delta\right). \tag{7.5}$$

For rotations, we are concerned with angular displacements ϕ instead of linear displacements x, and we replace mass m with moment of inertia I, and force F with torque τ. For rotations of a rigid body about a fixed axis the equation of motion becomes

$$\tau_{\text{net}} = I\alpha, \tag{7.6}$$

or

$$\tau_{\text{net}} = I\frac{d^2\phi}{dt^2}. \tag{7.7}$$

If the torque is linear in ϕ, and acts to return the body to its equilibrium position, then this equation can be rearranged to give

$$I\frac{d^2\phi}{dt^2} = -\kappa\phi, \tag{7.8}$$

where κ is just the proportionality constant between torque and angular displacement. Comparing this equation to Eq. (7.4) we see that the solution is

$$\phi(t) = A\cos\left(\sqrt{\frac{\kappa}{I}}\,t + \delta\right). \tag{7.9}$$

In this lab, you will determine an expression for the torsion constant κ in terms of the physical properties of your apparatus, you will measure the period of the "empty" trifilar pendulum, and you will combine these results to determine a value for I_0, the moment of inertia of the disk. You will compare this to a calculation of the moment of inertia of the disk based on Eq. (7.3) and measurements of the physical properties of the plate. You will then place additional solid bodies on the rotating plate and repeat the process to determine the moments of inertia for these bodies.

Procedure

1. For small angular displacements ϕ, the restoring torque provided by the strings on the disk is approximately proportional to ϕ. Consider how the strings exert torques on the plate, and show that

$$\kappa = \frac{MgR^2}{L},$$

   where M is the total mass suspended by the strings, L is the length of the strings, g is the gravitational field strength. and R is the perpendicular distance from the center of rotation to the point of application of the force resulting in the torque.

2. Measure the period of small oscillations of your plate. From this measurement and your expression for κ, determine an experimental value for the moment of inertia of the plate.

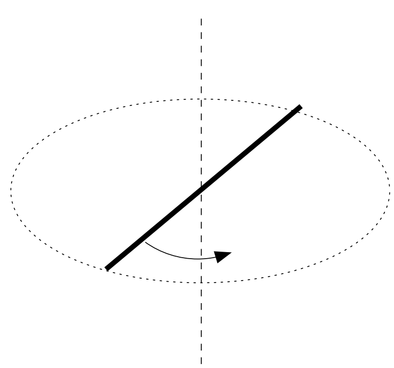
3. Determine the uncertainty in your this value for the moment of inertia of your plate. Smart use of Python can simplify this process.

4. Use Eq. (7.3) to calculate a theoretical formula giving the moment of inertia of your metal plate. To get a numerical value for the moment of inertia, you will have to determine the values for physical properties of the plate.

5. Determine an uncertainty in this second value for the moment of inertia of the plate.

6. Compare your two values of the moment of inertia of the plate.

7. Determine a theoretical formula for the moment of inertia of a thin rod about its center. Measure the physical properties of one of the rods in the lab and determine a theoretical value for the moment of inertia of the rod.



8. Measure the period of oscillation of the combined plate plus rod. From this

measurement determine an experimental value for the moment of inertia of the rod. Compare your experimental result with the value obtained in part 7.

9. Calculate the moment of inertia of a solid sphere starting from Eq.(7.3).

10. Measure the period of oscillation of the combined plate plus sphere. From this measurement determine an experimental value for the moment of inertia of the sphere. Compare your experimental result with the value obtained in part 9.

Notes to Instructors

1. .

Lab 8

Gyroscopes and Rotational Motion

PHYS 221, Fall 2019

Introduction

During this lab, you will become familiar with the basic principles underlying the motion of the gyroscope.

The Gyroscope

The gyroscope is an example of the application of Newton's second law for rotation,

$$\frac{d\mathbf{L}}{dt} = \sum_i \boldsymbol{\tau}_i, \tag{8.1}$$

where $\mathbf{L}$ is the angular momentum vector and the $\boldsymbol{\tau}_i$'s are the (vector) torques.

The components of the gyroscope you will be using consist of a flywheel (which can spin), a support base, and two movable counterweights attached to the rod on which the flywheel spins.

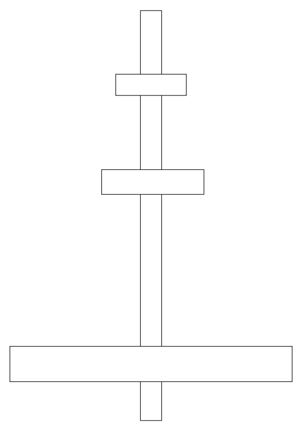
1. Without spinning the flywheel, use the counterweights to balance the gyroscope. Draw the free–body diagram associated with the balanced gyroscope; be sure to label all the forces present.

2. What is the net torque on the system in the balanced condition? Write down an expression for the net torque acting on the system in terms of masses and distances. What is the magnitude of this net torque in the balanced condition?

3. While holding the gyroscope in the balanced conditions, set the flywheel spinning and then release it. Describe what happens. Is this consistent with Newton's second law?

4. Stop the flywheel and return the gyroscope to its balanced, equilibrium position. Now move the heavy counterweight away from the equilibrium position by an amount Δr (a few centimeters will do). Show that the magnitude of the new net torque is $\tau = m_{\text{counter}-\text{weight}} g \, \Delta r$.

5. The figure below shows a view of the gyroscope in equilibrium looking down from above. Indicate the new position of the heavy counterweight and the direction you plan to spin the flywheel.



6. Make a vector diagram showing the instanteous $\mathbf{L}$ and the instantaneous $\boldsymbol{\tau}$.

7. Predict the magnitude of the angular velocity of the precession in terms of measurable quantities, and then predict the period of precession.

8. Start the flywheel spinning, and "play" with the apparatus.

9. With the flywheel spinning, get the gyroscope precessing evenly. Determine ω_{spin} using the tachometer and/or strobe light, and determine $T_{\text{precession}}$ with a stop watch.

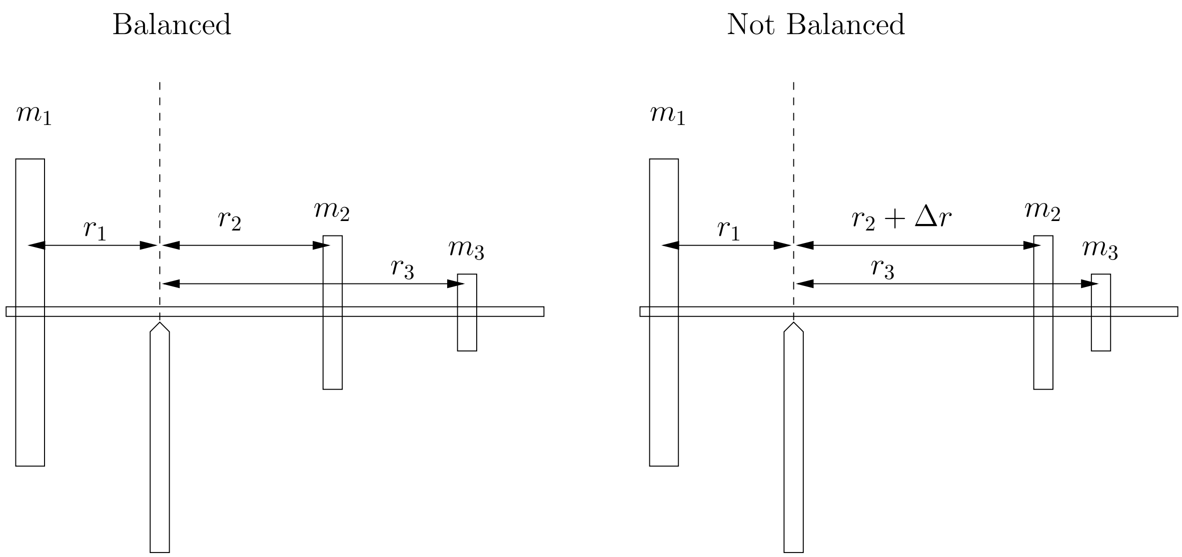
10. Make any other physical measurements necessary to compare your predicted period with your observations.

Notes to Instructors

French works almost exclusively with angular momentum about the center of mass, writing

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau}. \tag{8.2}$$

This is a useful expression because it is valid even when the center of mass is accelerating. (French shows this on pp. 639–640.) Analysis of our lab gyroscopes is a little more difficult if you insist on calculating torques about the center of mass, but not too bad. I'll do two versions of the calculation of the torque.



Version A: Calculating torques about the center of mass.

Choosing the pivot point as the origin, the net torque for the balanced gyroscope is

$$\tau_{\text{net}} = -m_1 r_1 g + m_2 r_2 g + m_3 r_3 g = 0, \tag{8.3}$$

and the center of mass is at the pivot. After moving mass 2 by an amount Δr the center of mass moves from the origin to

$$r_{\text{c.m.}} = \frac{-m_1 r_1 + m_2(r_2 + \Delta r) + m_3 r_3}{m_1 + m_2 + m_3} = \frac{m_2 \Delta r}{m_1 + m_2 + m_3}. \tag{8.4}$$

The net torque about the new center of mass must include the torque due to the force of support at the pivot point because it no longer acts through the center of mass:

$$\begin{aligned}\tau_{\text{net}} &= -m_1(r_1 + r_{\text{c.m.}})g + m_2(r_2 + \Delta r - r_{\text{c.m.}})g + m_3(r_3 - r_{\text{c.m.}})g + F_{\text{support}} r_{\text{c.m.}} \\ &= (-m_1 r_1 + m_2 r_2 + m_3 r_3)g - (m_1 + m_2 + m_3) r_{\text{c.m.}} + m_2 \Delta r g + F_{\text{support}} r_{\text{c.m.}}.\end{aligned}$$

The first term in the previous expression is zero, and the second and fourth terms cancel, leaving

$$\tau_{\text{net}} = m_2 \Delta r g. \tag{8.5}$$

Version B: Calculating torques about the pivot point.

Because the pivot point is not accelerating it's ok to use the expression

$$\frac{d\mathbf{L}}{dt} = \boldsymbol{\tau} \tag{8.6}$$

where the angular momentum and torques are not restricted to being calculated about the center of mass. In this case it is easy to show that in the unbalanced condition the net torque is simply

$$\begin{aligned}\tau_{\text{net}} &= -m_1 r_1 g + m_2(r_2 + \Delta r)g + m_3 r_3 g \\ &= (-m_1 r_1 g + m_2 r_2 g + m_3 r_3 g) + m_2 \Delta r g \\ &= m_2 \Delta r g \end{aligned} \tag{8.7}$$

Lab 9

Driven, Damped, Harmonic Oscillations (McAllister Apparatus)

PHYS 221, Fall 2019

Experiment

The McAllister Apparatus is a realization of a driven, damped mechanical oscillator. In this lab we will make measurements on a damped, driven mechanical oscillator to demonstrate how the amplitude of oscillation and phase of oscillation depends on the frequency of the driving force.

1. In order to ensure that your equipment is set up correctly, perform the following steps:

   (a) Turn on the power supply voltage to +12 Volts.

   (b) On the McAllister control box, set switch S1 in the "up" position and switch S2 in the "down" position.

   (c) Use the oscilloscope to view the pulses that are sent to the digital *stepper* motor, and measure the period and/or the frequency of the pulses. The black knob on the control box adjusts the pulse period.

   (d) For each pulse sent to the stepper motor, the motor turns by $1.8°$ (or $\pi/100\,\mathrm{rad}$. Therefore from measuring the pulse period P, you should be able to determine the oscillation frequency f (or period T) of the rotor.

Use a stop watch to measure the oscillation period for one setting, and confirm that this you are getting the right frequency (or period) from the oscilloscope.

2. Choose a few frequencies at random, and observe the amplitude of the oscillation after several cycles. Try to identify the approximate frequency at which resonance occurs. Make a qualitative sketch (by hand) of the amplitude of the oscillation *vs.* drive frequency, and phase of the oscillation (relative to the drive) *vs.* drive frequency. At low frequencies, the response and drive oscillations should be in phase with each other, and at high frequencies, they should be out of phase. Can you see this phase shift? Write a couple of sentences in your lab notebook about your observations.

3. Take careful data of both the amplitude and phase angle *vs.* the angular frequency of the driving force f. Estimating the phase lag of the oscillation is a little tougher than looking at variations in amplitude, and may be worth a few additional observations near frequencies of particular interest. Note that the LED flashes once per cycle, the rail holding the LED can be shifted up and down, and the phase at which the LED flashes can be adjusted using the rotating bar on the protractor. You should set it up so that at small ω it reads 0 degrees and the LED flashes each time the mass passes through its equilibrium point. It is also worth a few comments in your lab notebook on how well you could make this estimate.

4. Make sure you measure the amplitude of oscillations as $\omega \to 0$. To do this you can simply turn off the motor and look at the maximum and minimum values of the displacement when you turn the rotor by hand.

5. Make a rough measurement of the damping constant γ by observing the decay of free oscillations of your oscillator. One good choice would be to measure the time it takes for a free oscillation to decay in amplitude by a factor of 2.

6. Measure the period of oscillations of the mass when it is not in water. From this you can determine the natural frequency of the oscillator in the absence of damping.

7. Make a plot of your data of amplitude versus frequency.

8. Define a python function that returns the amplitude as a function of driving frequency.

9. Compare your results with the theory by displaying your amplitude data and the theoretical function on the same graph. Adjust the damping parameter γ to get the best correspondence between your data and theory.

10. Perform steps similar to (7, 8, 9) to examine the phase lag ϕ in your experiment.

Solution of the Equation of Motion

Consider Newton's second law for a driven damped oscillator:

$$\begin{aligned} m\frac{d^2x}{dt^2} &= F_{\mathrm{net}} \\ &= F_{\mathrm{spring}} + F_{\mathrm{damp}} + F_{\mathrm{drive}}. \end{aligned} \tag{9.1}$$

If we assume a sinusoidal driving term, a damping that is proportional to velocity, and a linear spring, this equation can be written

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} + F_0\cos\omega t. \tag{9.2}$$

Dividing through by m gives:

$$\frac{d^2x}{dt^2} = -\omega_0^2 x - \gamma\frac{dx}{dt} + \frac{F_0}{m}\cos\omega t, \tag{9.3}$$

where $\omega_0^2 \equiv k/m$ and $\gamma \equiv b/m$. Now let's let turn this into a differential equation for a *complex* function of time z, whose real part will be the actual physical displacement x. The complex form of Eq. (9.3) is

$$\frac{d^2z}{dt^2} = -\omega_0^2 z - \gamma\frac{dz}{dt} + \frac{F_0}{m}e^{i\omega t}, \tag{9.4}$$

From your experimentation with the McAllister apparatus you should see that the steady state motion is sinusoidal at the *drive* frequency ω, although the oscillation may not be in phase with the drive. This suggests that we look for solutions of the form

$$z = Ae^{i(\omega t-\phi)}. \tag{9.5}$$

The constant A will be the amplitude of the motion, and the constant ϕ will give the phase *lag* of the displacement with respect to the driving force. (If the displacement is in phase with the driving force, then $\phi = 0$.)

After substituting the assumed form of the displacement, from Eq. (9.5), into the differential equation of motion, Eq. (9.4), one can take all time derivatives, solve for A, and find the amplitude A and phase lag ϕ. You should find an amplitude that depends on the drive frequency:

$$A(\omega) = \frac{F_0/m}{\left[\left(\omega_0^2 - \omega^2\right)^2 + \left(\gamma\omega\right)^2\right]^{1/2}} \tag{9.6}$$

and a phase lag that looks like:

$$\phi(\omega) = \arctan\frac{\gamma\omega}{(\omega_0^2 - \omega^2)} \tag{9.7}$$

Compare your results with your theoretical predictions.

Notes to Instructors (with some history)

For several years there was a systematic that we hadn't accounted for. Measuring the spring constant and mass, or equivalently the period when the mass is hanging in air, gave an ω_0 which was large compared to the position of the measured resonance peak. I thought that this might have been due to the effect of the buoyant force changing with y, but I wasn't sure. In 2001 M.L. considered making hangars with "all" mass at the bottom.

In 2007 M.L. came up with a different attack on this very real systematic problem. Dave Schoepf and M.L. recalled that in the "old days" we used masses with very different vanes: we both remember thin brass (?) vanes on thin rods. In recent years we have been using 1/4 inch threaded rod with thick Lucite crosses at the bottom. Dave Vayda doesn't remember the former apparatus the way we do, but I'm pretty sure he's the one who constructed the current Lucite crosses. (The drawings in French make it look like thin vanes too with all the mass at the top.) In 2006 I had two students do a project to improve the results. They made the bobber out of a coat hanger with clay mass on the coat hanger ABOVE the surface of the water. (They also embedded some extra steel balls in the clay for extra mass.) This greatly reduces the damping, and the majority of the mass is out of the water at all times. They got very good results, but because the Q is so large it was difficult to take data with our existing stepper motor apparatus. It's not only tough to map out the resonance curve because it's so narrow, but the mass jumps out of the liquid if your don't have just the right mass, drive amplitude, etc.

In 2007 I had everybody use thrown-together bobs made out of thin aluminum rod with shim stock vanes duct-taped to the bottom. I also duct-taped some standard disk weights near the top of the aluminum rods. In spite of the crude apparatus we got much, much better quantitative agreement than in recent years. Is this the result of reducing the effect of time-varying buoyant forces, or the result of going to a regime of damping proportional to v rather than v^2? I'm not sure, but I think it's the latter. (Shear forces vs. end-on forces?) The 2006 students did a little bit of numerical investigation of v^2 damping, but it wasn't very complete or documented.

In 2008 Dave Vayda made some new bobs out of thin aluminum rod with thin plastic vanes. M.L. taped $4 \times 20\,\mathrm{g}$ disk weights at the top of the rod. This gives a period of free oscillation in air of about $1\,\mathrm{s}$. We used water as the damping fluid. If you use the "middle" hole for the drive, the resonance is big enough to be easy to measure, but not so big that the bob jumps out of the water.

Measuring γ: You can make a crude estimate of the damping constant γ by measuring the time $T_{1/2}$ it takes undriven oscillations to damp out to half their original value:

$$e^{-T_{1/2}\gamma/2} = \frac{1}{2} \quad \Rightarrow \quad \gamma = \frac{2\ln 2}{T_{1/2}}.$$

The damping time is about 4 s, which gives $\gamma \simeq 0.35$.

Determining F_0/m: The amplitude of the oscillations for very small frequencies goes to

$$A(0) = \frac{F_0/m}{\omega_0^2}.$$

This amplitude can be measured with the drive off, and simple rotation of the rotor by hand. This gives $F_0/m = A(0)\,\omega_0^2$.

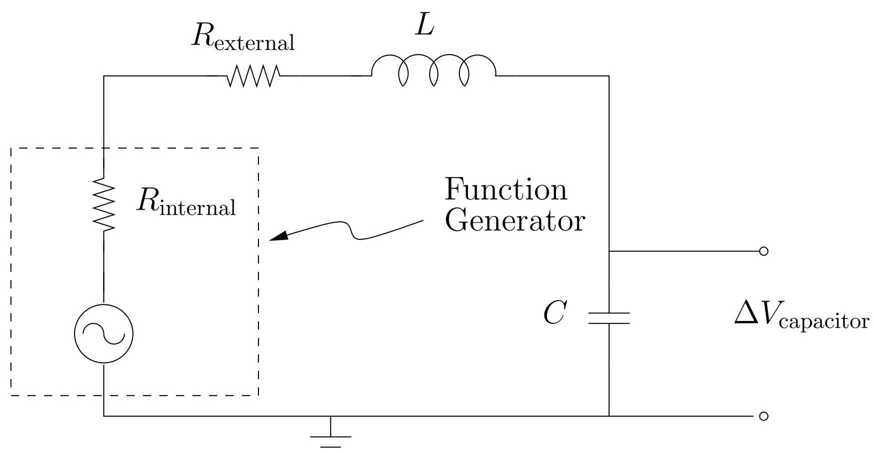
NOTE: The simple complex number exercises may not be necessary, depending on how this lab is scheduled relative to the lecture coverage.

Lab 10

Driven, Damped, Electrical Oscillations

PHYS 221, Fall 2019

This lab is the electrical analog of the driven, damped, mechanical oscillations lab that you recently completed. In place of the McAllister machine you will construct a simple circuit consisting of a function generator (voltage source), a resistor, a capacitor, and an inductor (coil). The circuit is illustrated below:



We will get an "equation of motion" for the charge q on the capacitor by using Kirchoff's Law: the sum of the voltage drops around a loop must be zero. If we call the current in our circuit i, the voltage drop across the resistors is

$$\Delta V_{\mathrm{resistor}} = iR_{\mathrm{total}}, \tag{10.1}$$

where R is the resistance in Ohms (Ω), the voltage drop across the capacitor is

$$\Delta V_{\text{capacitor}} = \frac{q}{C}, \tag{10.2}$$

where q is the charge on the capacitor in Coulomb's (C), and the voltage drop across the inductor (coil) is

$$\Delta V_{\text{inductor}} = L\frac{di}{dt}, \tag{10.3}$$

where $i = dq/dt$ is the current in ampere's (A), or Coulomb/s, and L is the inductance in henries (H). The function generator produces a sinusoidal voltage

$$V_{\text{generator}} = V_0 \cos \omega t. \tag{10.4}$$

Adding up the voltage differences around the circuit gives

$$\begin{aligned} 0 &= V_{\text{generator}} - \Delta V_{\text{resistor}} - \Delta V_{\text{inductor}} - \Delta V_{\text{capacitor}} \\ &= V_0 \cos \omega t - iR - L\frac{di}{dt} - \frac{q}{C} \end{aligned} \tag{10.5}$$

Using the fact that $i = dq/dt$, and rearranging, gives

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V_0 \cos \omega t. \tag{10.6}$$

This should look familiar in form! The equation of motion of the driven damped *mechanical* oscillator that you have been working with recently is

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F_0 \cos \omega t. \tag{10.7}$$

You should reason by analogy to convince yourself that the amplitude and the phase (relative to the driving voltage) of the voltage across the capacitor are given by the following equations:

$$\Delta V_{\text{capacitor}} = \frac{V_0 \omega_0^2}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + \omega^2 \gamma^2}}, \tag{10.8}$$

and

$$\tan \delta = \frac{\gamma \omega}{\omega_0^2 - \omega^2}, \tag{10.9}$$

where

$$\omega_0 = \frac{1}{\sqrt{LC}} \qquad \text{and} \qquad \gamma = \frac{R}{L}. \tag{10.10}$$

You can determine the values of L and C using the hand-held meters in the lab. The value of R is the sum of the resistance you put in the circuit, plus about $50\,\Omega$ of internal resistance in the function generator and the inductor. Convince yourself that V_0 can be determined from the amplitude of $\Delta V_{\text{capicitor}}$ when the freqency is very small.

Procedure

1. Set up the circuit illustrated in the figure. Use the oscilloscope to observe simultaneously the voltage across the terminals of the function generator and the voltage drop across the capacitor.

2. Set the function generator to give sinusoidal oscillations. Observe the amplitude and phase of the voltage drop across the capacitor as you vary the frequency of the oscillator. Find the resonant frequency that gives the maximum voltage. Before taking any careful data make a qualitative sketch of the amplitude of the voltage across the capacitor as a function of the frequency of the function generator. Make a second *qualitative* sketch of the phase lag of the voltage across the capacitor (relative to the drive voltage from the function generator) as a function of frequency.

3. Make careful measurements of the amplitude and phase lag of the voltage across the capacitor. Start with a frequency near resonance, and take a couple of data points above resonance, and a couple of data points below resonance. There are features of your oscilloscope that will make data collection pretty easy.

4. Plot your preliminary amplitude data *vs.* frequency. Also plot the theoretical function giving the amplitude as a function of frequency. Adjust your damping constant to give the best fit. Is the damping constant about what you expect it to be?

5. Plot your preliminary phase data *vs.* frequency. Also plot the theoretical function giving the phase as a function of frequency.

6. Once your confident that you are correctly taking data that makes your theoretical predictions, take additional data to see how well the theoretical predictions match over a wide range of frequencies.

7. Now set up the function generator to give you a relatively low frequency *square wave*. You should see damped *free* oscillations of your electrical system that are analogous to those you would get if you gave a mechanical oscillator a single kick. Measure the frequency of the oscillations you see, and make an approximation of the damping constant γ from the half-life of the decay of the oscillations. The frequency you measure should be close to the frequency you predict from theory, and close to the frequency at which you got the largest $\Delta V_{\text{capacitor}}$ when the circuit was driven with a sine wave. Verify these assertions.

Notes to Instructors

1. Components:
   - Inductors: Faded green potted inductors; nominally 0.015 H (actually closer to 0.013 H.
   - Capacitors: 0.022 μF ceramic capacitors give good results. The capacitance can, of course, vary quite a bit from this nominal value. Using cheap ceramics is good because you can demonstrate temperature dependence: you set the circuit to resonance and then blow on the capacitor or hold it between your fingers and watch the drift away from resonance.
   - Resistor: It's fine to rely on internal resistance of signal generator and other internal resistances.
2. **Make sure amplitude of drive is low.** It's easy to drive things so hard that elements become nonlinear; you can even hear audio oscillations in inductor at resonance with large drive amplitudes.
3. There is always the perennial choice of which voltage to measure. In PHYS 235 we often measure voltage across the resistor because the current is the easiest thing to derive: $i = v/Z$. As of 2007 this 221 lab is written to measure the current across the capacitor. This is the closest analog to the physical displacement measured in the McAllister machine.
4. In 2007 M.L. began to realize how bad standard inductors can be. The faded green potted inductors are the highest Q inductors we have. The inductors in the silver "cans" have about the same inductance, but much lower Q. The low Q affects the result very significantly if you're looking for quantitative agreement. Modeling including the effect of the imperfect inductors (with frequency-dependent properties) continues.
5. We have hand-held inductance meters now that effectively replace the old GR Bridge.

Lab 11

Coupled Oscillations and Normal Modes

PHYS 221, Fall 2019

Introduction

In previous classes and labs you have studied the motion of single, isolated oscillators. In this lab you are going to investigate *coupled* oscillators. Coupled oscillators can have very complex motions, but there exist states of the system, called *normal modes*, in which the motion is quite simple. A normal mode of a system of oscillators is a state in which all parts of the system oscillate at the same frequency.

In this experiment you will investigate the oscillations of rods hung as pendulums from a common steel band. The pendulums are coupled because they "feel" the presence of their nearest neighbors via the twisting of the steel band.

Theory

Each swinging rod obeys Newton's second law in the form

$$\sum_i \tau_i = I_{\text{rod}} \frac{d^2\theta}{dt^2}, \tag{11.1}$$

where θ is the angular displacement of the rod from equilibrium (*i.e.*, vertical) and $\sum \tau_i$ is the net torque on the rod. The net torque acting on a given rod is the sum of three parts:

1. the gravitational torque,
2. a torque caused by any twist in the segment of the band immediately to the left of the rod, and
3. a torque caused by any twist in the segment of the band immediately to the right of the rod.

The torque produced by the steel band is proportional to the angle of the twist of the band in a given section divided by the length of that section of the band:

$$|\tau_{\text{band}}| = \kappa \frac{|\Delta\theta|}{s}, \tag{11.2}$$

where $\Delta\theta$ is the twist angle of a section of band of length s, and κ is the torsion constant of the steel band that you will determine experimentally.

For two pendulums suspended symmetrically with respect to the center of the band, there are two simple normal modes:

$$\textbf{Mode I:} \quad \theta_1(t) = \theta_2(t) \tag{11.3}$$

$$\textbf{Mode II:} \quad \theta_1(t) = -\theta_2(t) \tag{11.4}$$

For three pendulums suspended at $D/4$, $D/2$, and $3D/4$ there are three normal modes:

$$\textbf{Mode I:} \quad \theta_1(t) = -\theta_3(t); \quad \theta_2(t) = 0 \tag{11.5}$$

$$\textbf{Mode II:} \quad \theta_1(t) = \theta_3(t) = \frac{\theta_2(t)}{\sqrt{2}} \tag{11.6}$$

$$\textbf{Mode III:} \quad \theta_1(t) = \theta_3(t) = -\frac{\theta_2(t)}{\sqrt{2}} \tag{11.7}$$

Procedure

Theory

1. Show that the moment of inertia of a rod of mass M and length L which rotates around one end is given by

$$I_{\text{rod}} = \frac{1}{3}ML^2.$$

2. Show that the gravitational torque acting on a single rod which is displaced from equilibrium by an angle θ is given by

$$\tau_{\text{grav}} = -\frac{1}{2}MgL\sin\theta \simeq -\frac{1}{2}MgL\theta.$$

3. Show that the torque due to the steel band on a single rod suspended from the center of the band is given by

$$\tau_{\text{band}} = -\frac{4\kappa}{D}\theta,$$

   where D is the total length of the band.

4. Write down the equation of motion for a single rod placed at the center of the band, a distance $D/2$ from either end. Find a theoretical expression for the angular frequency of oscillation.

5. Find an algebraic expression for the torsion constant κ in terms of the quantities M, I, L, D, and T (the period of oscillation of the single bar).

6. Consider the case of two rods suspended at $2D/5$ and $3D/5$. Write down the equation of motion for each of the two rods.

7. For the case of two bars, use the information given in Eq. (11.3) and your equation(s) of motion to find a theoretical expression for the frequency (and period) of normal mode I.

8. For the case of two bars, use the information given in Eq. (11.4) and your equation(s) of motion to find a theoretical expression for the frequency (and period) of normal mode II.

9. Consider the case of three rods suspended at $D/4$, $D/2$, $3D/4$. Write down an equation of motion for one of the three rods.

10. Find a theoretical expression for one of the normal mode frequencies for the three-rod case.

Experiment

1. Measure the mass of the rods, the length of the rods, and the length of the steel band.

2. Measure the period of the single-rod system. From this period and your expression for the frequency, calculate the torsion constant κ.

3. Measure the periods of both normal modes of the two-rod system.

4. Measure the periods of all three normal modes of the three-rod system.
5. Compare your results for the normal mode frequencies with your theoretical predictions.

Notes to Instructors

- The "small" setups are probably better than the large, floppy, "old" setups. The good ones are made of two-by-fours, and use the shorter rods.
- The brass rods are stored in the green cabinet with drawers at the back of the electronics lab. (One of the lower drawers.)
- Rocking of the apparatus can be a problem. (The modes of oscillation include more than just the rods!) Make sure to stabilize the apparatus with wooden wedges from the drawer with the rods and/or C-clamps.
- Be careful about confusion between angular frequency ω and the "other" $\omega = \dot{\theta}$.
- This worked well in 2001 because it was preceded by a lecture on the three-spring–two-mass problem in class. We discussed the equations of motion for the individual masses, the idea of normal modes, and then we "guessed" the motion, and solved for the normal mode frequencies. The context of the lab problems is different enough that it wasn't too repetitive. I also previewed some stuff about rotational dynamics that we will be getting to shortly.
- Be sure to tighten the band for each set of equipment. This makes a remarkable difference in the success of the lab.
- It's important to keep amplitude of oscillations *very* small. There are three reasons:
    1. usual small angle approximation,
    2. this keeps excitation of rocking modes of entire apparatus small, and
    3. the band doesn't loosen with small amplitude oscillations.
- M.L. got a torsion constant of period of $T = 23.7/20 = 1.185\,\mathrm{s}$ which gives a torsion constant of $\kappa = 0.0593$. The torsion constant can be as much as a factor of 2 higher with a very tight band. In the two-bar case this leads to predicted normal mode periods of $T_{\mathrm{I}} = 1.212\,\mathrm{s}$ and $T_{\mathrm{II}} = 1.061\,\mathrm{s}$, to be compared to the measured values of 1.2075 and 1.0405 s. Students like the fact that things appear to agree so well — it looks like results agree at the level of 1-2%. This isn't really a fair comparison though. It might be more honest to look at the effect of the coupling between the masses, and this results in a difference in normal mode frequencies. This agreement is at the 10% level. really quite so good

"Theory" Answers:

1.

$$\begin{aligned} I_{\text{rod}} = \int r^2\, dm &= \int_0^L x^2 \frac{M}{L}\, dx \\ &= \frac{1}{3} M L^2 \end{aligned}$$

2. Use pivot point as origin. Force of support contributes no torque and the gravitational force acts through the center of mass at the mid-point of the rod.

3.

$$\begin{aligned} \tau_{\text{band}} &= \tau_{\text{left}} + \tau_{\text{right}} \\ &= -\kappa \frac{\theta}{D/2} - \kappa \frac{\theta}{D/2} \\ &= -\frac{4\kappa}{D}\theta. \end{aligned}$$

4.

$$\begin{aligned} I \frac{d^2\theta}{dt^2} &= \tau_{\text{net}} \\ &= \tau_{\text{grav}} + \tau_{\text{band}} \\ &= -\frac{1}{2} M g L \theta - \frac{4\kappa}{D}\theta \\ &= -\left(\frac{1}{2} M g L + \frac{4\kappa}{D}\right)\theta. \end{aligned}$$

Therefore

$$\omega^2 = \frac{1}{I}\left(\frac{1}{2} M g L + \frac{4\kappa}{D}\right).$$

5. Solving for the torsion constant gives

$$\begin{aligned} \kappa &= \frac{D}{4}\left(-\frac{1}{2} M g L + I\omega^2\right) \\ &= \frac{D}{4}\left(-\frac{1}{2} M g L + I\frac{4\pi^2}{T^2}\right) \end{aligned}$$

6. Equation of motion for bar 1:

$$\begin{aligned} I\frac{d^2\theta_1}{dt^2} &= \tau_{\text{net}} \\ &= \tau_{\text{grav}} + \tau_{\text{band left}} + \tau_{\text{band right}} \\ &= -\frac{1}{2}MgL\theta_1 - \kappa\frac{\theta_1}{2D/5}\theta_1 + \kappa\frac{(\theta_2-\theta_1)}{D/5} \end{aligned}$$

Equation of motion for bar 2:

$$\begin{aligned} I\frac{d^2\theta_2}{dt^2} &= \tau_{\text{net}} \\ &= \tau_{\text{grav}} + \tau_{\text{band left}} + \tau_{\text{band right}} \\ &= -\frac{1}{2}MgL\theta_2 - \kappa\frac{(\theta_2-\theta_1)}{D/5} - \kappa\frac{\theta_1}{2D/5}\theta_1 \end{aligned}$$

7. In mode I we have $\theta_1 = \theta_2$, so the equation of motion of bar 1 becomes

$$I\frac{d^2\theta_1}{dt^2} = -\left(\frac{MgL}{2} + \frac{5\kappa}{2D}\right)\theta_1$$

and therefore

$$\omega_I^2 = \frac{1}{I}\left(\frac{Mgl}{2} + \frac{5\kappa}{2D}\right).$$

Using bar 2 gives the same frequency (as it must in a normal mode).

8. In mode II we have $\theta_1 = -\theta_2$, so the equation of motion of bar 1 becomes

$$\begin{aligned} I\frac{d^2\theta_1}{dt^2} &= -\frac{1}{2}MgL\theta_1 - \kappa\frac{\theta_1}{2D/5}\theta_1 - \kappa\frac{2\theta_1}{D/5} \\ &= -\left(\frac{MgL}{2} + \frac{25\kappa}{2D}\right)\theta_1 \end{aligned}$$

and therefore

$$\omega_{II}^2 = \frac{1}{I}\left(\frac{MgL}{2} + \frac{25\kappa}{2D}\right).$$

Using bar 2 gives the same frequency (as it must in a normal mode).

9. Equation of motion of bar 1:

$$\begin{aligned} I\frac{d^2\theta_1}{dt^2} &= \tau_{\text{net}} \\ &= \tau_{\text{grav}} + \tau_{\text{band left}} + \tau_{\text{band right}} \\ &= -\frac{1}{2}MgL\theta_1 - \kappa\frac{\theta_1}{D/4} + \kappa\frac{(\theta_2-\theta_1)}{D/4} \end{aligned}$$

Equation of motion for bar 2:

$$\begin{aligned} I\frac{d^2\theta_2}{dt^2} &= \tau_{\text{net}} \\ &= \tau_{\text{grav}} + \tau_{\text{band left}} + \tau_{\text{band right}} \\ &= -\frac{1}{2}MgL\theta_2 - \kappa\frac{(\theta_2-\theta_1)}{D/4} - \kappa\frac{(\theta_2-\theta_3)}{D/4} \end{aligned}$$

Equation of motion of bar 3:

$$\begin{aligned} I\frac{d^2\theta_3}{dt^2} &= \tau_{\text{net}} \\ &= \tau_{\text{grav}} + \tau_{\text{band left}} + \tau_{\text{band right}} \\ &= -\frac{1}{2}MgL\theta_3 - \kappa\frac{(\theta_3-\theta_2)}{D/4} - \kappa\frac{\theta_3}{D/4} \end{aligned}$$

10. **Mode I:** $\theta_2 = 0$ and $\theta_1 = -\theta_3$
    Equation of motion of bar 1 is

$$\begin{aligned} I\frac{d^2\theta_1}{dt^2} &= -\frac{1}{2}MgL\theta_1 - \kappa\frac{\theta_1}{D/4} + \kappa\frac{(0-\theta_1)}{D/4} \\ &= -\left(\frac{MgL}{2} + \frac{8\kappa}{D}\right)\theta_1 \end{aligned}$$

giving

$$\omega_{\text{I}}^2 = \frac{1}{I}\left(\frac{MgL}{2} + \frac{8\kappa}{D}\right).$$

Mode II: $\theta_1 = \theta_3$ and $\theta_2 = \sqrt{2}\theta_1$
Equation of motion of bar 1 is

$$\begin{aligned} I\frac{d^2\theta_1}{dt^2} &= -\frac{1}{2}MgL\theta_1 - \kappa\frac{\theta_1}{D/4} + \kappa\frac{(\sqrt{2}\theta_1-\theta_1)}{D/4} \\ &= -\left(\frac{MgL}{2} + \frac{4(2-\sqrt{2})\kappa}{D}\right)\theta_1 \end{aligned}$$

giving

$$\omega_{\text{II}}^2 = \frac{1}{I}\left(\frac{MgL}{2} + \frac{4(2-\sqrt{2})\kappa}{D}\right).$$

Checking this result with equation of motion for bar 2:

$$\begin{aligned} I\frac{d^2\theta_2}{dt^2} &= -\frac{1}{2}MgL\theta_2 - \kappa\frac{(\theta_2 - \theta_2/\sqrt{2})}{D/4} - \kappa\frac{(\theta_2 - \theta_2/\sqrt{2})}{D/4} \\ &= -\left(\frac{MgL}{2} - \frac{4(2-\sqrt{2})\kappa}{D}\right)\theta_2 \end{aligned}$$

verifying that

$$\omega_{\mathrm{II}}^2 = \frac{1}{I}\left(\frac{MgL}{2} + \frac{4(2-\sqrt{2})\kappa}{D}\right).$$

Mode III: $\theta_1 = \theta_3$ and $\theta_2 = -\sqrt{2}\theta_1$
Equation of motion of bar 1 is

$$\begin{aligned} I\frac{d^2\theta_1}{dt^2} &= -\frac{1}{2}MgL\theta_1 - \kappa\frac{\theta_1}{D/4} + \kappa\frac{(-\sqrt{2}\theta_1 - \theta_1)}{D/4} \\ &= -\left(\frac{MgL}{2} + \frac{4(2+\sqrt{2})\kappa}{D}\right)\theta_1 \end{aligned}$$

giving

$$\omega_{\mathrm{III}}^2 = \frac{1}{I}\left(\frac{MgL}{2} + \frac{4(2+\sqrt{2})\kappa}{D}\right).$$

Checking this result with equation of motion for bar 2:

$$\begin{aligned} I\frac{d^2\theta_2}{dt^2} &= -\frac{1}{2}MgL\theta_2 - \kappa\frac{(\theta_2 + \theta_2/\sqrt{2})}{D/4} - \kappa\frac{(\theta_2 + \theta_2/\sqrt{2})}{D/4} \\ &= -\left(\frac{MgL}{2} - \frac{4(2+\sqrt{2})\kappa}{D}\right)\theta_2 \end{aligned}$$

verifying that

$$\omega_{\mathrm{II}}^2 = \frac{1}{I}\left(\frac{MgL}{2} + \frac{4(2+\sqrt{2})\kappa}{D}\right).$$