A UNIFICATION OF PERMUTATION PATTERNS RELATED TO SCHUBERT VARIETIES

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Abstract

We prove new connections between permutation patterns and singularities of Schubert varieties, by giving a new characterization of factorial and Gorenstein varieties in terms of so called *bivincular patterns*. These are generalizations of classical patterns where conditions are placed on the location of an occurrence in a permutation, as well as on the values in the occurrence. This clarifies what happens when the requirement of smoothness is weakened to factoriality and further to Gorensteinness, extending work of Bousquet-Mélou and Butler (2007), and Woo and Yong (2006). We also prove results that translate some known patterns in the literature into bivincular patterns.

Nous démontrons de nouveaux liens entre les motifs de permutation et les singularités des variétés de Schubert, par la méthode de donner une nouvelle caractérisation des variétés factorielles et de Gorenstein par rapport à les *motifs bivinculaires*. Ces motifs sont généralisations des motifs classiques où des conditions se posent sur la position d'une occurrence dans une permutation, aussi bien que sur les valeurs qui se présentent dans l'occurrence. Ceci éclaircit les phénomènes où la condition de nonsingularité s'affaiblit á factorialité et même à Gorensteinité, et augmente les travaux de Bousquet-Mélou et Butler (2007), et de Woo et Yong (2006). Nous démontrons également des résultats qui traduisent quelques motifs connus en la littérature en motifs bivinculaires.

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Introduction

The goal of this project was to unify descriptions of permutation patterns related to Schubert varieties in the complete flag manifold. Before we state our results we review some definitions. The results obtained can be summarized in the table below.

X_π is	The permutation π avoids the patterns
smooth	2143 and 1324
factorial	$2\underline{14}3$ and 1324
Gorenstein	$\begin{smallmatrix}12\overline{34}5\\3\underline{15}24\end{smallmatrix}$, $\begin{smallmatrix}1\overline{23}45\\24\underline{15}3\end{smallmatrix}$; and satisfies a condition on descents involving two infinite families of bivincular patterns

Complete flags and Schubert cells

We will only consider complete flags in $\mathbb{C}^m$ so we will simply refer to them as *flags*. A flag is a sequence of vector-subspaces of $\mathbb{C}^m$

$$E_\bullet = (E_1 \subset E_2 \subset \cdots \subset E_m = \mathbb{C}^m),$$

with the property that $\dim E_i = i$. The set of all such flags is called the (*complete*) *flag manifold*, and denoted by $F\ell(\mathbb{C}^m)$. We want to consider special subsets of this flag manifold called *Schubert cells*.

If we choose a basis $f_1, f_2, \ldots, f_m$, for $\mathbb{C}^m$ then we can fix a *reference flag*

$$F_\bullet = (F_1 \subset F_2 \subset \cdots \subset F_m)$$

such that F_i is spanned by the first i basis vectors. Using this reference flag and a permutation π in $\mathfrak{S}_m$ we can define the *Schubert cell* $X_\pi^\circ \subseteq F\ell(\mathbb{C}^m)$ which contains the flags $E_\bullet$ such that

$$\dim(E_p \cap F_q) = \#\{i \leq p \,|\, \pi(i) \leq q\},$$

for $1 \leq p, q \leq m$.

Example (A Schubert cell in $F\ell(\mathbb{C}^3)$)**.** *Let $\pi = 231$. Then $\pi(1) = 2$, $\pi(2) = 3$ and $\pi(3) = 1$. The conditions for the Schubert cell X_{231}°*

$$\dim(E_p \cap F_q) = \#\{i \leq p \,|\, \pi(i) \leq q\},$$

become

	$p=1$	$p=2$	$p=3$	
$q=1$	0	0	1	*E_1, E_2 intersect F_1 in a point*
$q=2$	1	1	2	*$E_1 \subset F_2$, $E_2 \cap F_2 = E_1$*
$q=3$	1	2	3	

We should also notice that this Schubert cell can be described with the matrix

$$\begin{pmatrix} * & 1 & 0 \\ * & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

so $X_{231}^\circ \cong \mathbb{C}^2$.

Given a Schubert cell X_π° we define the *Schubert variety* as the closure

$$X_\pi = \overline{X_\pi^\circ},$$

in the Zariski topology.

Pattern avoidance

Classical patterns

An *occurrence* (or *embedding*) of a pattern p in a permutation π is classically defined as a subsequence in π, of the same length as p, whose letters are in the same relative order (with respect to size) as those in p. For example, the pattern 123 corresponds to a increasing subsequence of three letters in a permutation. If we use the notation 1_π to denote the first, 2_π for the second and 3_π for the third letter in an occurrence, then we are simply requiring that $1_\pi < 2_\pi < 3_\pi$. If a permutation has no occurrence of a pattern p we say that π *avoids* p.

Example. *The permutation* 32415 *contains two occurrences of the pattern* 123 *corresponding to the sub-words* 345 *and* 245*. It avoids the pattern* 132*.*

Vincular patterns

In a *vincular pattern* (also called a *generalized pattern*, *Babson-Steingrímsson pattern* or *dashed pattern*), two adjacent letters may or may not be underlined. If they are underlined it means that the corresponding letters in the permutation π must be adjacent.

Example. *The permutation* 32415 *contains one occurrence of the pattern* $\underline{12}3$ *corresponding to the sub-word* 245*. It avoids the pattern* $1\underline{23}$*. The permutation* $\pi = 324615$ *has one occurrence of the pattern* 2143*, namely the sub-word* 3265*, but no occurrence of* $2\underline{14}3$*, since* 2 *and* 6 *are not adjacent in* π*.*

These types of patterns have been studied sporadically for a very long time but were not defined in full generality until Babson and Steingrímsson (2000).

Bivincular patterns

This notion was generalized further in Bousquet-Mélou et al. (2010): In a *bivincular pattern* we are also allowed to put restrictions on the values that occur in an embedding of a pattern. We use two-line notation to describe these patterns. If there is a line over the letters i, $i+1$ in the top row, it means that the corresponding letters in an occurrence must be adjacent in values. This is best described by an example:

Example. *An occurrence of the pattern* $\genfrac{}{}{0pt}{}{\overline{12}3}{123}$ *in a permutation* π *is an increasing subsequence of three letters, such that the second one is larger than the first by exactly* 1*, or more simply* $2_\pi = 1_\pi + 1$*. The permutation* 32415 *contains one occurrence of this bivincular pattern corresponding to the sub-word* 345*. This is also an occurrence of* $\genfrac{}{}{0pt}{}{\overline{123}}{123}$*. The permutation avoids the bivincular pattern* $\genfrac{}{}{0pt}{}{\overline{123}}{1\underline{23}}$*.*

Barred patterns
We will only consider a single pattern of this type, but the general definition is easily inferred from this special case. We say that a permutation π *avoids the barred pattern* $21\bar{3}54$ if π avoids the pattern 2143 (corresponding to the unbarred elements) *except* where that pattern is a part of the pattern 21354. This notation for barred patterns was introduced by West (1990). It turns out that avoiding this barred pattern is equivalent to avoiding $2\underline{14}3$.

Example. *The permutation* $\pi = 425761$ *avoids the barred pattern* $21\bar{3}54$ *since the unique occurrence of* 2143*, as the sub-word* 4276*, is contained in the sub-word* 42576 *which is an occurrence of* 21354*.*

Bruhat restricted patterns
Here we recall the definition of Bruhat restricted patterns from Woo and Yong (2006). First we need the *Bruhat order* on permutations in S_n, defined as follows: Given integers $i < j$ in $[\![1, n]\!]$ and a permutation $\pi \in S_n$ we define $w(i \leftrightarrow j)$ as the permutation that we get from π by swapping $\pi(i)$ and $\pi(j)$. For example $24153(1 \leftrightarrow 4) = 54123$. We then say that $\pi(i \leftrightarrow j)$ *covers* π if $\pi(i) < \pi(j)$ and for every k with $i < k < j$ we have either $\pi(k) < \pi(i)$ or $\pi(k) > \pi(j)$. We then define the Bruhat order as the transitive closure of the above covering relation. We see that in our example above that $24153(1 \leftrightarrow 4)$ does not cover 24153 since we have $\pi(2) = 4$. Now, given a pattern p with a set of transpositions $\mathcal{T} = \{(i_\ell \leftrightarrow j_\ell)\}$ we say that a permutation π *contains* $(p, \mathcal{T})$, or that π *contains the Bruhat restricted pattern* p, if $\mathcal{T}$ is understood from the context, if there is an embedding of p in π such that if any of the transpositions in $\mathcal{T}$ are carried out on the embedding the resulting permutation covers π.
We should note that Bruhat restricted patterns were further generalized to *intervals of patterns* in Woo and Yong (2008). We will not consider this generalization here.
Below we will show how these three types of patterns are related to one another.

Summary of results

Recall the theorem of Ryan (1987), Wolper (1989) and Lakshmibai and Sandhya (1990) that the Schubert variety X_π is non-singular (or smooth) if and only if π avoids the patterns 1324 and 2143. Saying that the variety X_π is non-singular means that every local ring is regular. A weakening of this condition is the requirement that every local ring only be a unique factorization domain; a variety satisfying this is a *factorial* variety.

Bousquet-Mélou and Butler (2007) proved a conjecture stated by Yong and Woo (Bousquet-Mélou et al., 2005, Personal communication) that factorial Schubert varieties are those that correspond to permutations avoiding 1324 and bar-avoiding $21\bar{3}54$. In the terminology of Woo and Yong (2006) the bar-avoidance of the latter pattern corresponds to avoiding 2143 with Bruhat condition $(1 \leftrightarrow 4)$, or equivalently, interval avoiding $[2413, 2143]$ in the terminology of Woo and Yong (2008). However, as remarked by Steingrímsson (2007), bar-avoiding $21\bar{3}54$ is equivalent to avoiding the vincular pattern $2\underline{14}3$. If we summarize this in terms of vincular patterns a striking thing becomes apparent; see the lines corresponding to smoothness and factoriality in the table below.

X_π is	The permutation π avoids the patterns
smooth	2143 and 1324
factorial	$2\underline{14}3$ and 1324
Gorenstein	$\begin{smallmatrix} 12\overline{34}5 \\ 3\underline{15}24 \end{smallmatrix}$, $\begin{smallmatrix} 1\overline{23}45 \\ 24\underline{15}3 \end{smallmatrix}$; and satisfies a condition on descents involving two infinite families of bivincular patterns

We see that requiring 1 and 4 to be adjacent in the first pattern (and thus turning it into a vincular pattern) corresponds to weakening smoothness to factoriality.

A further weakening is to only require that the local rings of X_π be Gorenstein local rings, in which case we say that X_π is a *Gorenstein* variety. Woo and Yong (2006) showed that X_π is Gorenstein if and only if it avoids two patterns with two Bruhat restrictions each, as well as satisfying a certain condition on descents. We will translate their results into avoidance of bivincular patterns; see below and the line corresponding to Gorensteinness in the table above. Essentially the two bivincular patterns that are shown there are certain *upgrades* of the pattern $2\underline{14}3$ while the avoidance of the pattern 1324 is weakened to avoidance of two infinite families of bivincular patterns.

Idea behind the proof of the Gorenstein case

Woo and Yong (2006) classify those permutations π that correspond to Gorenstein Schubert varieties X_w. They do this using embeddings of patterns with Bruhat restrictions, which we have described above, and with a certain condition on the associated Grassmannian permutations of w, which we will describe presently:

First, a *descent* in a permutation π is an integer d such that $\pi(d) > \pi(d+1)$. A *Grassmannian permutation* is a permutation with a unique descent. Given any permutation π we can associate a Grassmannian permutation to each of its descents, as follows: Given a particular descent d of π we construct the sub-word $\gamma_d(\pi)$ by concatenating the right-to-left minima of the segment strictly to the left of $d+1$ with the left-to-right maxima of the segment strictly to the right of d. More intuitively we start with the descent $\pi(d)\pi(d+1)$ and enlarge it to the left by adding decreasing elements without creating another descent and similarly enlarge it to the right by adding increasing elements without creating another descent. We then denote the *flattening* (or *standardization*) of $\gamma_d(\pi)$ by $\tilde{\gamma}_d(\pi)$, which is the unique permutation whose letters are in the same relative order as $\gamma_d(\pi)$.

Example. *Consider the permutation* $\pi = 11|6|12|94153728|10$ *where we have used the symbol* $|$ *to separate two digit numbers from other numbers. For the descent at* $d = 4$ *we get* $\gamma_4(\pi) = 694578|10$ *and* $\tilde{\gamma}_4(\pi) = 3612457$.

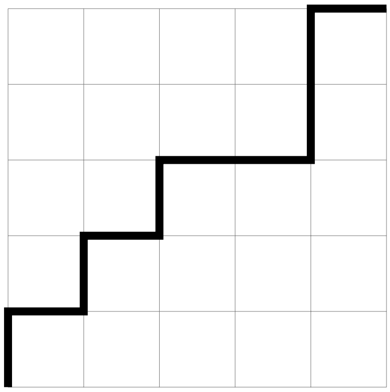
Now, given a Grassmannian permutation π in S_n with its unique descent at d we construct its *associated partition* $\lambda(\pi)$ as the partition inside a bounding box $d \times (n-d)$, with d rows and $n-d$ columns, whose lower border is the lattice path that starts at the lower left corner of the bounding box and whose i-th step, for $i \in [\![1, n]\!]$, is vertical if i is weakly to the left of the position d, and horizontal otherwise. We are interested in the *inner corner distance* of this partition, *i.e.*, for every inner corner we add its distance from the left side and the distance from the top of the bounding box. If all these inner corner distances are the same then the inner corners all lie on the same anti-diagonal.

In Theorem 1 of Woo and Yong (2006) they show that a permutation $\pi \in S_n$ corresponds to a Gorenstein Schubert variety X_π if and only if

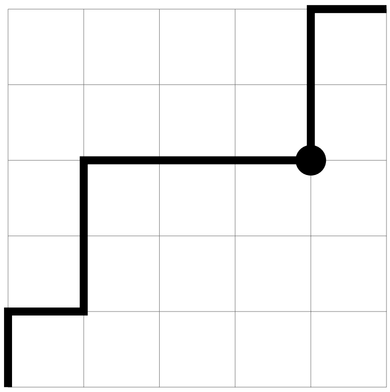
(1) for each descent d of π, $\lambda(\tilde{\gamma}_d(\pi))$ has all of its inner corners on the same anti-diagonal; and
(2) the permutation π avoids both 31524 and 24153 with Bruhat restrictions $\{(1 \leftrightarrow 5), (2 \leftrightarrow 3)\}$ and $\{(1 \leftrightarrow 5), (3 \leftrightarrow 4)\}$, respectively.

Translating the second condition into bivincular patterns is not difficult and we omit the details here.

Translating condition 1 of Theorem 1 of Woo and Yong (2006) is a bit more work. First, let's see some examples of these partitions. Here is the partition corresponding to $\pi = 13589 \downarrow 2467|10$.



Then we have the partition corresponding to $\pi = 13489 \downarrow 2567|10$:



The failure of this condition is easily seen to be equivalent to some partition λ of an associated Grassmannian permutation $\tilde{\gamma}_d(\pi)$ having an outer corner that is either "too deep" or "too wide", as in the marked outer corner in the second figure above. This outer corner comes from the subsequence $134\mathbf{89} \downarrow \mathbf{2567}|\mathbf{10}$. This is an occurrence of the bivincular pattern

$$\begin{matrix}1\overline{234567}\\1562347\end{matrix}\cdot$$

More generally, given a Grassmannian permutation π and an outer corner of $\lambda(\pi)$, we say that it is *too wide* if the distance upward from it to the next inner corner is smaller than the distance to the left from it to the next inner corner. Conversely we say that an outer corner is *too deep* if the distance upward from it to the next inner corner is larger than the distance to the left from it to the next inner corner. We say that an outer corner is*unbalanced* if it is either too wide or too deep. We say that an outer corner is *balanced* if it is not *unbalanced.*
If a permutation has an associated Grassmannian permutation with an outer corner that is too wide we say that the permutation itself is *too wide* and similarly for *too deep.* If the permutation is either too wide or too deep we say that it is *unbalanced*, otherwise it is *balanced.* It is time to see some examples.
These properties of Grassmannian permutations can be detected with bivincular patterns, as in the example above.

Lemma. *Let π be a Grassmannian permutation.*

(1) The permutation π is too wide if and only if it contains at least one of the bivincular patterns from the infinite family

$$\mathcal{F} = \left(\begin{smallmatrix} 1\overline{2345} \\ 14235 \end{smallmatrix}, \begin{smallmatrix} 1\overline{234567} \\ 1562347 \end{smallmatrix}, \begin{smallmatrix} 1\overline{23456789} \\ 167823459 \end{smallmatrix}, \dots \right).$$

The general member of this family is of the form

$$\begin{smallmatrix} 1\overline{2\cdots\cdots\cdots k} \\ 1\ell{+}1\cdots2\cdots\ell k \end{smallmatrix},$$

where $\ell = (k-3)/2$.

(2) The permutation π is too deep if and only if it contains at least one of the bivincular patterns from the infinite family

$$\mathcal{G} = \left(\begin{smallmatrix} \overline{1234}5 \\ 13425 \end{smallmatrix}, \begin{smallmatrix} \overline{123456}7 \\ 1456237 \end{smallmatrix}, \begin{smallmatrix} \overline{12345678}9 \\ 156782349 \end{smallmatrix}, \dots \right).$$

The general member of this family is of the form

$$\begin{smallmatrix} \overline{12\cdots\cdots\cdots}k \\ 1\ell{+}1\cdots2\cdots\ell k \end{smallmatrix},$$

where $\ell = (k-1)/2$.

Proof. We only consider part 1, as part 2 is proved analogously. Assume that π is a Grassmannian permutation that is too wide, so it has an outer corner that is too wide. Let ℓ be the distance from this outer corner to the next inner corner above. Then the distance from this outer corner to the next inner corner to the left is at least $\ell + 1$. This allows us to construct an increasing sequence t of length ℓ in π, starting at a distance at least two to the right of the descent. We can also choose t so that every element in it is adjacent both in location and values. Similarly we can construct an increasing sequence s of length ℓ in π, located strictly to the left of the descent. We can also choose s so that every element in it is adjacent both in location and values. This produces the required member of the family $\mathcal{F}$.
Conversely, assume π contains a particular member of the family $\mathcal{F}$. Then π clearly has at least one outer corner that is too wide. □

It should be noticed that these two infinite families are obtained from one another by reverse complement.
We have now shown that

Proposition. *A permutation π is balanced if and only if every associated Grassmannian permutation avoids every bivincular pattern in the two infinite families $\mathcal{F}$ and $\mathcal{G}$ in the lemma above.* □

This gives us:

Theorem. *Let $\pi \in S_n$. The Schubert variety X_π is Gorenstein if and only if*

(1) π is balanced; and

(2) the permutation π avoids the bivincular patterns

$$\begin{smallmatrix}12\overline{34}5\\3\underline{15}24\end{smallmatrix} \text{ and } \begin{smallmatrix}1\overline{23}45\\24\underline{15}3\end{smallmatrix}.$$

$\square$

I should note that with the descriptions of factorial and Gorenstein Schubert varieties given above it is easy to verify that smoothness implies factoriality implies Gorensteinness.

Future problems

(1) *Further simplification*: The description of Gorenstein Schubert varieties would be nicer if one could avoid considering the associated Grassmannian permutations altogether. This is indeed possible if one resorts to a further generalization of patterns, so-called *grid patterns*. The details of how this is achieved will be in the journal version of this paper.

(2) *In between factorial and Gorenstein*: What happens when we underline different letters in the original patterns 1324 and 2143? Would this correspond to new singularity properties of Schubert varieties.

(3) *Pushing pattern avoidance further I*: How far can the connection between the singularity properties of Schubert varieties and bivincular patterns be taken? In particular, what can be applied to other flag varieties than $\mathrm{Flags}_n(\mathbb{C})$? One concrete question that remains unanswered (Billey (2009)) is how to describe the singular loci of Schubert varieties of type B; extending results of Billey and Warrington (2003) for type A.

(4) *Pushing pattern avoidance further II*: A permutation is *vexillary* if it avoids the pattern 2143, Lascoux and Schützenberger (1985), or, equivalently, if its Stanley symmetric function equals the Schur function for a particular shape. Billey and Lam (1998) define vexillary permutations in the hyperoctahedral group B_n as those permutations that bar-avoid 18 different patterns. It would be interesting to find a simpler description in terms of bivincular patterns, or grid patterns.

(5) *Kazhdan-Lusztig polynomials*: Finally Kazhdan-Lusztig polynomials are combinatorial objects that have been intensively studied and are notoriously difficult to compute. For a survey, see Brenti (2002/04). They have many relations to Schubert varieties and in particular, Kazhdan and Lusztig (1979, 1980) showed that X_π is rationally smooth at ρ if and only if the Kazhdan-Lusztig polynomial $P_{\rho,\pi}(q)$ is identically equal to 1. See also Billey and Mitchell (2008). It would be interesting to see if bivincular pattern avoidance could be used to gather more combinatorial information on these polynomials.

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