Nonzero coefficients in restrictions and tensor products of supercharacters of $U_n(q)$

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Supercharacters

A supercharacter theory of a group G is a pair $(\mathcal{K}, \mathcal{X})$ where $\mathcal{K}$ is a partition of G and $\mathcal{X}$ is a partition of the irreducible characters of G such that

- $|\mathcal{K}| = |\mathcal{X}|$
- $\{1\} \in \mathcal{K}$ and $1\!\!1 \in \mathcal{X}$
- $K \in \mathcal{K}$ is a union of conjugacy classes
- For $S \in \mathcal{X}$, $\sum_{\chi \in S} \chi(1)\chi$ is constant on the parts of $\mathcal{K}$.

We call the parts of $\mathcal{K}$ superclasses and the characters $\{\sum_{\chi\in S}\chi(1)\chi \mid S \in \mathcal{X}\}$ supercharacters.

Examples

- $\mathcal{K} = \{\{1\}, G - \{1\}\}$, $\mathcal{X} = \{1\!\!1, \chi_{\mathbb{C}G} - 1\!\!1\}$
- $\mathcal{K} = \left\{\begin{matrix}\text{conjugacy}\\ \text{classes}\end{matrix}\right\}$, $\mathcal{X} = \left\{\begin{matrix}\text{irreducible}\\ \text{characters}\end{matrix}\right\}$

Main Example

$$U_n = \left\{ \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix} \;\middle|\; * \in \mathbb{F}_q \right\}$$

For this poster $q = 2$

$$\cup\!\!\!\mid$$

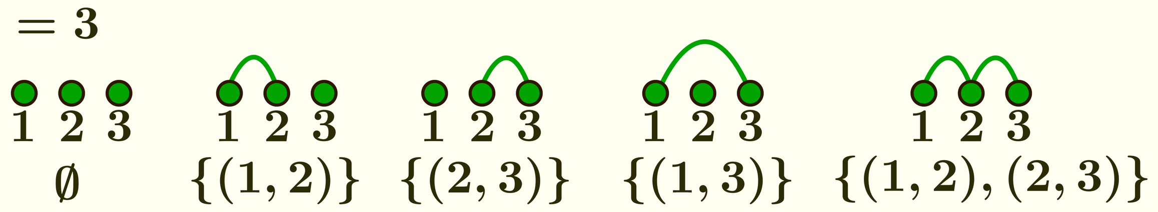
$$U_A = \{u \in U_n \mid u_{ij} \neq 0, i < j \text{ implies } i, j \in A\}$$

where $A \subseteq \{1, 2, \ldots, n\}$ $(U_A \cong U_{|A|})$

Superclasses

$$\mathcal{K} = 1 + U_n \backslash (U_n - 1)/U_n \longleftrightarrow \left\{\begin{matrix}\text{Sets } \lambda \text{ of pairs } (i,j) \in [n] \times [n] \\ \text{such that } i < j \text{ and } (i,j),(k,l) \in \lambda \\ \text{implies } i \neq k \text{ and } j \neq l\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Set partitions} \\ \text{of } \{1,2,\ldots,n\}\end{matrix}\right\}$$

$n = 3$



Supercharacters

For λ a set partition, define

$$\chi^\lambda = \prod_{\overset{\frown}{i\ j} \in \lambda} \chi^{\overset{\frown}{i\ j}} \quad \text{where} \quad \chi^{\overset{\frown}{i\ l}}(\mu) = \begin{cases} \dfrac{q^{l-i-1}(-1)^{\#\{(i,l)\in\mu\}}}{q^{\#\{(j,k)\in\mu \mid i<j<k<l\}}}, & \text{if } i<j<l \text{ implies } (i,j),(j,l) \notin \mu, \\ 0, & \text{otherwise.} \end{cases}$$

Main Theorem

Theorem. For $A \subseteq \{1, 2, \ldots, n\}$, the inner product $\langle \chi^\lambda \otimes \chi^\mu, \chi^\nu \rangle \neq 0$ / $\langle \mathrm{Res}^{U_n}_{U_A}(\chi^\lambda), \chi^\nu \rangle \neq 0$ if and only if the bipartite graph $\Gamma^\nu_{\lambda\mu} = (V_{\text{green}} \cup V_{\text{red}}, E)$ / $\Gamma^\nu_{\lambda A} = (V_{\text{green}} \cup V_{\text{red}}, E)$ has a complete matching from V_{green} to V_{red}.

The bipartite graph construction