Boolean complexes and boolean numbers

Bridget Eileen Tenner

DePaul University

`bridget@math.depaul.edu` `math.depaul.edu/~bridget`

The Bruhat order gives a poset structure to any Coxeter group. The ideal of elements in this poset having boolean principal order ideals forms a simplicial poset. This simplicial poset defines the boolean complex for the group. In a Coxeter system of rank n, we show that the boolean complex is homotopy equivalent to a wedge of $(n-1)$-dimensional spheres. The number of these spheres is the boolean number, which can be computed inductively from the unlabeled Coxeter system, thus defining a graph invariant. For certain families of graphs, the boolean numbers have intriguing combinatorial properties. This work involves joint efforts with Claesson, Kitaev, and Ragnarsson.

(W, S) is a finitely generated Coxeter system with the (strong) Bruhat ordering.

Elements of W with boolean principal order ideals are boolean.

They form a simplicial subposet $\mathbb{B}(W, S)$ called the boolean ideal.

The boolean complex of (W, S) is the regular cell complex $\Delta(W, S)$ whose face poset is the simplicial poset $\mathbb{B}(W, S)$.

Lemma. *An element of W is boolean if and only if it has no repeated letters in its reduced words.*

$\Longrightarrow \Delta(W, S)$ is pure, and each maximal face has dimension $|S| - 1$.

We study the homotopy type of the geometric realization $|\Delta(W, S)|$.

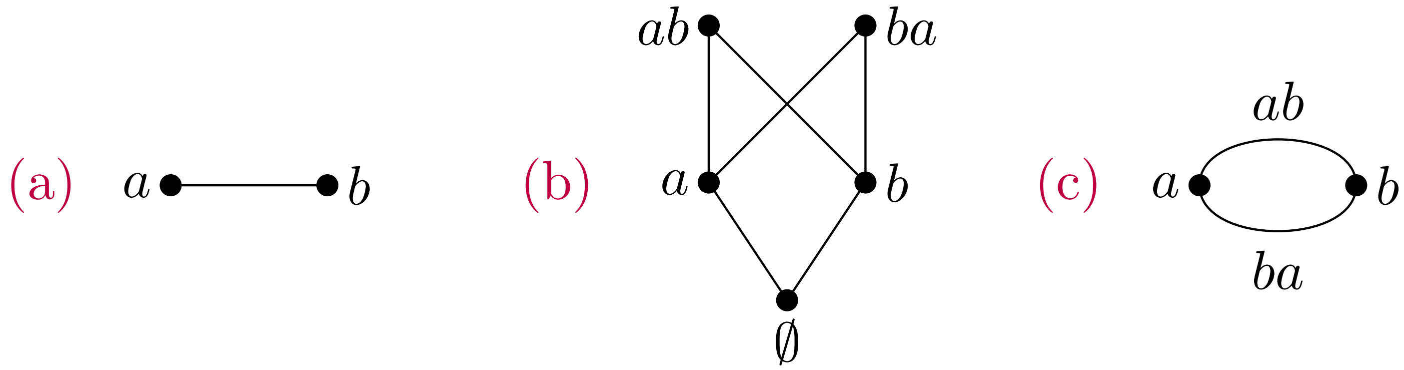
Since boolean elements have no repeated letters in their reduced decompositions, the only relation we care about in (W, S) is whether two letters commute.

Thus we can look at the unlabeled Coxeter graph $G = G(W, S)$.

Or rather ... any finite simple graph G.

$\mathbb{B}(G)$ and $\Delta(G)$ are analogous: isomorphic to $\mathbb{B}(W, S)$ and $\Delta(W, S)$.

Example.



(a) The graph K_2. (b) The poset $\mathbb{B}(K_2)$. (c) The boolean complex $\Delta(K_2)$, where $|\Delta(K_2)|$ is homotopy equivalent to S^1.

The unlabeled Coxeter graphs of the Coxeter groups $A_2, B_2/C_2, G_2$ and $I_2(m)$ are all the same as K_2.

For a finite graph G, $|G|$ is the number of vertices in G.

Let G be a finite simple graph and e an edge in G.

- Deletion: $G - e$ is the graph obtained by deleting the edge e.
- Simple contraction: G/e is the graph obtained by contracting the edge e and then removing all loops and redundant edges.
- Extraction: $G - [e]$ is the graph obtained by removing the edge e and its incident vertices.

For $n \geq 1$, δ_n is the graph consisting of n disconnected vertices.

$\simeq$ denotes homotopy equivalence.

Theorem ([RT]). *For every nonempty, finite simple graph G, there is an integer $\beta(G)$ so that*

$$|\Delta(G)| \simeq \beta(G) \cdot S^{|G|-1}.$$

Moreover, $\beta(G)$ can be computed using the recursive formula

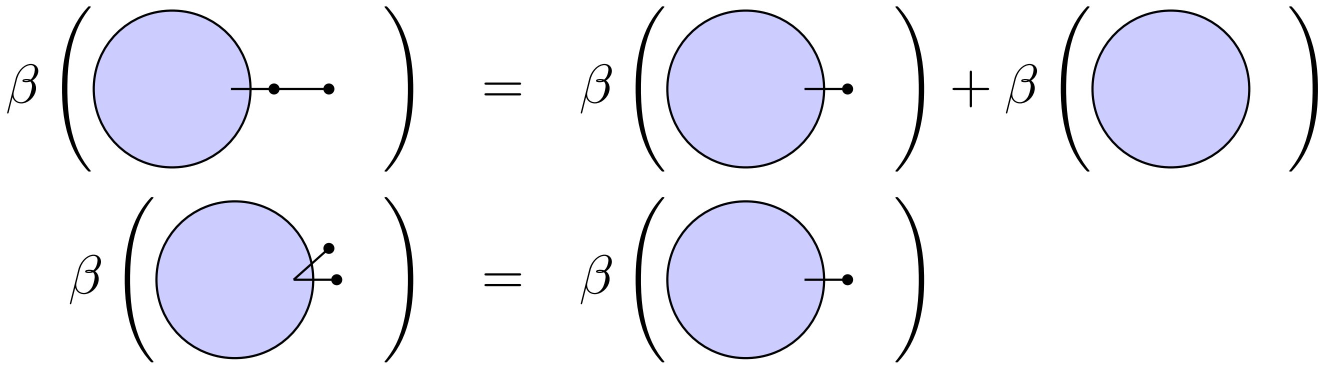
$$\beta(G) = \beta(G - e) + \beta(G/e) + \beta(G - [e]),$$

if e is an edge in G with $G - [e] \neq \emptyset$, with initial conditions

$$\beta(K_2) = 1 \text{ and } \beta(\delta_n) = 0.$$

Proposition ([RT]). $\Delta(H_1 \sqcup H_2) = \Delta(H_1) * \Delta(H_2)$ *where $*$ denotes simplicial join, and $\beta(H_1 \sqcup H_2) = \beta(H_1)\beta(H_2)$.*

Corollary. *For a vertex of degree one, computing $\beta(G)$ is easy:*



Corollary. *G has an isolated vertex if and only if $\beta(G) = 0$.*

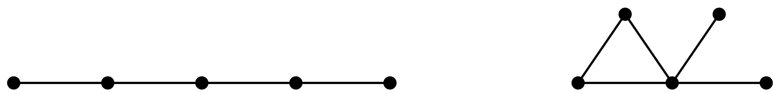
Corollary. *For $n \geq 1$, $\beta(K_n)$ is the number of derangements of $[n]$.*

Corollary. *$\beta(G - e) = \beta(G)$ if and only if G has an isolated vertex (so $\beta(G) = \beta(G - e) = 0$). Otherwise $\beta(G - e) < \beta(G)$.*

The function β, from graphs to $\mathbb{N}$, is a graph invariant.

We can look at its enumerative properties ...

Example. β is not injective: the two graphs below each have boolean number 3, and thus are each $\simeq S^4 \vee S^4 \vee S^4$.



Example. β is not surjective onto an interval: no graph on 4 vertices has boolean number 4, although $\beta(\text{4-cycle}) = 5$.

In other enumerative directions, there are families of graphs whose boolean numbers give well-known sequences.

- graphs with disjoint vertices: always 0
- paths: Fibonacci numbers
- complete graphs: derangement numbers

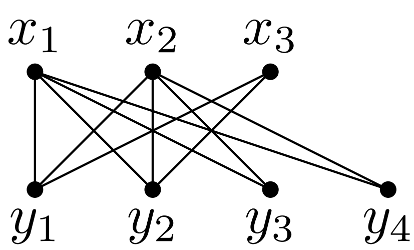
Other such families involve Ferrers graphs . . .

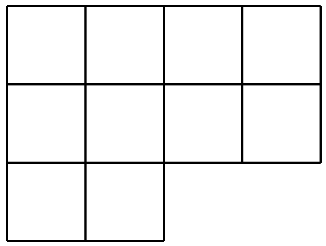
To any Ferrers shape, or Young shape or partition, is a corresponding Ferrers graph:

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition, where $\lambda_1 \geq \cdots \geq \lambda_r \geq 0$.

The associated bipartite Ferrers graph has vertices $\{x_1, \ldots, x_r\} \sqcup \{y_1, \ldots, y_{\lambda_1}\}$, and edges $\big\{\{x_i, y_j\} : \lambda_i \geq j\big\}$.

Example. The Ferrers graph and shape for $\lambda = (4, 4, 2)$:





The Ferrers graph of an m-by-n rectangular shape is the complete bipartite graph $K_{m,n}$.

Computation of the boolean number of such a graph invokes the Stirling numbers of the second kind ...

Corollary ([CKRT])**.** *For* $m, n \geq 1$,

$$\beta(K_{m,n}) = \sum_{k=1}^{m} (-1)^{m-k} k! \left\{ \begin{matrix} m+1 \\ k+1 \end{matrix} \right\} k^n.$$

The median Genocchi number g_n is equal to the number of permutations of $2n$ letters having alternating excedances.

For $n \geq 1$, the staircase shape of height n is the Ferrers shape

$$\Sigma_n = (n, n-1, \ldots, 2, 1).$$

Let F_n denote the Ferrers graph for the Ferrers shape Σ_n.

Corollary ([CKRT]). *For $n \geq 1$, $\beta(F_n) = g_n$.*

References

[Bjö] A. Björner, Posets, regular CW complexes and Bruhat order, European J. Combin. 5 (1984), 7–16.

[BW] A. Björner and M. Wachs, On lexicographically shellable posets, Trans. Amer. Math. Soc. 277 (1983), 323–341.

[Bre] F. Brenti, A combinatorial formula for Kazhdan-Lusztig polynomials, Invent. Math. 118 (1994), 371–394.

[CKRT] A. Claesson, S. Kitaev, K. Ragnarsson, and B.E. Tenner, Boolean complexes for Ferrers graphs, preprint.

[ES] R. Ehrenborg and E. Steingrímsson, The excedance set of a permutation, Adv. Appl. Math. 24 (2000), 284–299.

[JW] J. Jonsson and V. Welker, Complexes of injective words and their commutation classes, Pacific J. Math. 243 (2009), 313–329.

[RT] K. Ragnarsson and B.E. Tenner, Homotopy type of the boolean complex of a Coxeter system, Adv. Math. 222 (2009), 409–430.

[RW] V. Reiner and P. Webb, The combinatorics of the bar resolution in group cohomology, J. Pure Appl. Algebra 190 (2004), 291–327.

[Ten] B.E. Tenner, Pattern avoidance and the Bruhat order, J. Combin. Theory, Ser. A 114 (2007), 888–905.