# Crossings and Nestings in Set Partitions of Classical Types

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## Abstract

We study bijections

$$\{\text{ Set partitions of type X }\} \xrightarrow{\sim} \{\text{ Set partitions of type X }\}$$

for $X \in \{A, B, C, D\}$, which preserve **openers** and **closers**.

In types $A$, $B$, and $C$, they interchange

- either the number of **crossings** and of **nestings**,
- or the cardinalities of a **maximal crossing** and of a **maximal nesting**.

In type $D$, the results are obtained only in the case of **non-crossing** and **non-nesting set partitions**.

In all types, we show in particular that **non-crossing** and a **non-nesting set partition** are essentially **uniquely determined** by its openers and closers.

## Set partitions for classical types (via intersection lattices)

- A set partition of type $A_{n-1}$ is a usual set partition of $[n]$:

$$\{\{1,4\},\{2,5,7,9\},\{3,6\},\{8\}\}.$$

- A set partition of type $B_n$ or $C_n$ is a set partition $\mathcal{B}$ of $[\pm n]$ such that

$$B \in \mathcal{B} \Leftrightarrow -B \in \mathcal{B},$$

having at most one block $B = -B$:

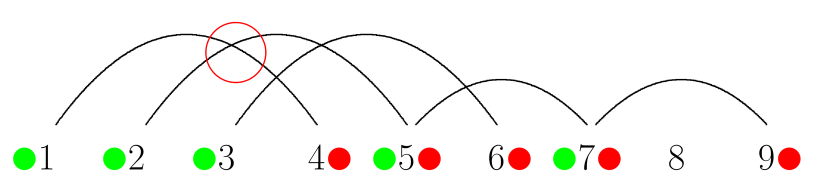
$$\{\{1,2,4,-1,-2,-4\},\{3,-5\},\{5,-3\}\}.$$

- A set partition of type $D_n$ is a set partition $\mathcal{B}$ of type $B_n$ for which the zero block (if present) is a single pair $\{i,-i\}$:

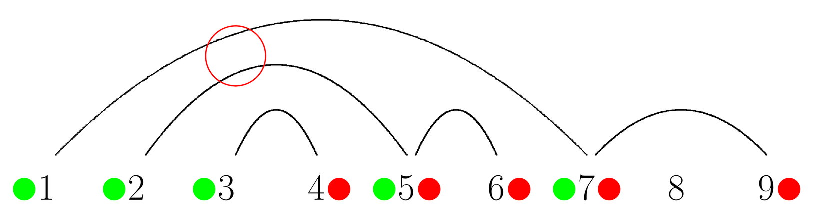
$$\{\{1,-2\},\{2,-1\},\{4,-4\},\{3,-5\},\{5,-3\}\}.$$

- A **nesting** in a set partition is a quadruple $(i < k < l < j)$ such that $i, j$ are contained in one block and $j, k$ in another (plus additional properties in types $B$ and $D$), where the **nesting order** is given by $1 < \ldots < n < -n \ldots < -1$.
- A **crossing** in a set partition is a quadruple $(i < k < j < l)$ such that $i, j$ are contained in one block and $j, k$ in another (plus additional properties in types $B$ and $D$), where the **crossing order** is given by $1 < \ldots < n < -1 \ldots < -n$.
- Non-maximal elements (in the **nesting order**) in blocks of a set partition are called **openers** and non-minimal elements are called **closers**.

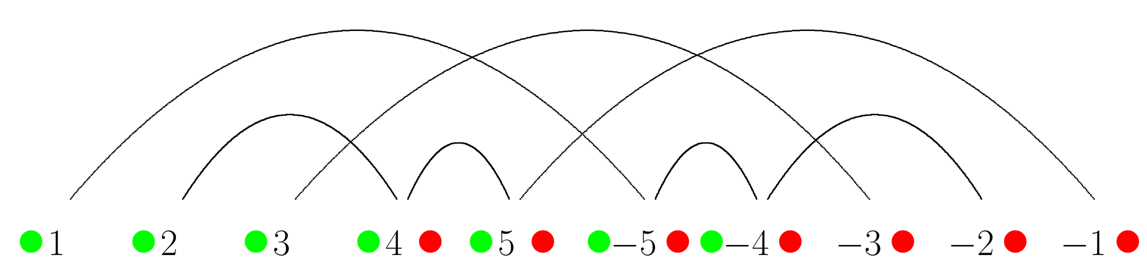
## Visualizations of set partitions



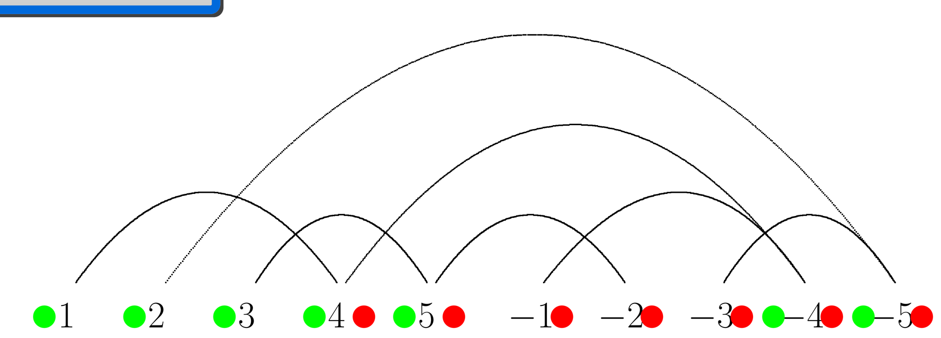

Non-nesting set partition $\{\{1,4\},\{2,5,7,9\},\{3,6\},\{8\}\}$




Non-crossing set partition $\{\{1,7,9\},\{2,5,6\},\{3,4\},\{8\}\}$




Nesting diagram of the type $C_5$ set partition
$\{\{1,-5,-4,-2\},\{2,4,5,-1\},\{3,-3\}\}$




Crossing diagram of the type $C_5$ set partition
$\{\{1,4,-4,-1\},\{2,-5,-3\},\{3,5,-2\}\}$

## Results in different types

**Theorem (Kasraoui, Zeng / Chen, Deng, Du, Stanley):**
In type $A$, there exist explicit bijections preserving openers and closers such that

- the number of crossings and nestings are interchanged, **or**
- the cardinality of a maximal crossing and of a maximal nesting are interchanged.

**Theorem:**
For non-crossing and non-nesting partitions, there exists a unique bijection preserving openers and closers.

**Theorem:**
In types $B$ and $C$, there exist explicit bijections preserving openers and closers such that

- the number of crossings and nestings are interchanged, **or**
- the cardinality of a maximal crossing and of a maximal nesting are interchanged.

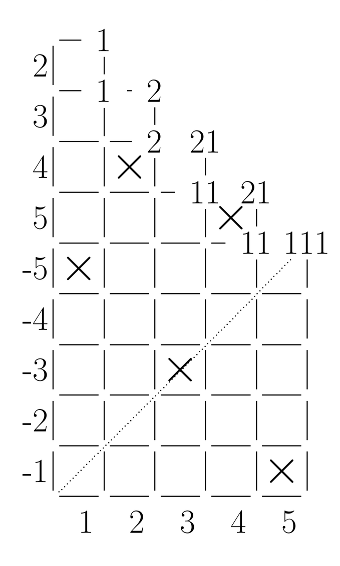
**Theorem:**
For non-crossing and non-nesting partitions, there exists a unique bijection preserving openers and closers.

**Theorem:**
In type $D$, there exist an explicit and essentially unique bijection between non-crossing and non-nesting partitions preserving openers and closers.

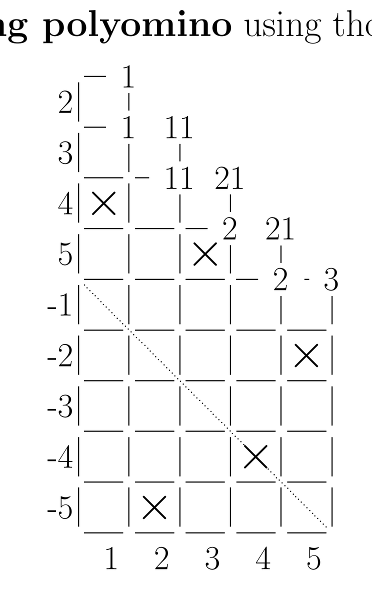
**Remark:**
In type $D$, there is a notion of non-crossing and non-nesting set partitions but we do not have a notion of crossings and nestings.

## Maximal crossings $\leftrightarrow$ maximal nestings in type $C$

- encode a type $C$ set partition as a $0-1$-filling of a **nesting polyomino**: $\{\{1,-5,-4,-2\},\{2,4,5,-1\},\{3,-3\}\}$



- nestings are then encoded as **strict north-east chains**
- note how the symmetry is reflected in the diagram
- label the boundary with partitions for growth diagrams,
- transpose all partitions,
- fill the **crossing polyomino** using those partitions:



- read off the resulting set partition using again the type $C$ symmetry:

$$\{\{1,4,-4,-1\},\{2,-5,-3\},\{3,5,-2\}\}$$

## Number of crossings $\leftrightarrow$ number of nestings

The bijection in type $C$ can be described as follows:

- go through all **positive closers** $c$ starting with 1,
  - denote by $\ell$ the number of **active openers** (openers which are not connected to closers left of $c$).
- if $c$ is connected to the $k$-th active opener, change this connection to the $(\ell - k)$-th active opener (this construction was explained by Kasraoui, Zeng);
- do the analogous construction on the **negative openers** and **active closers**;
- observe that two arcs connecting a positive opener with a negative closer cross if and only if they nest; thus all connections between positive openers and negative closers remain the same.

**Remark:**

- in type $B$, one can use a slight variation of both bijections;
- in type $D$, we obtaine the bijections only between non-nesting and non-crossing partitions.

## References

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