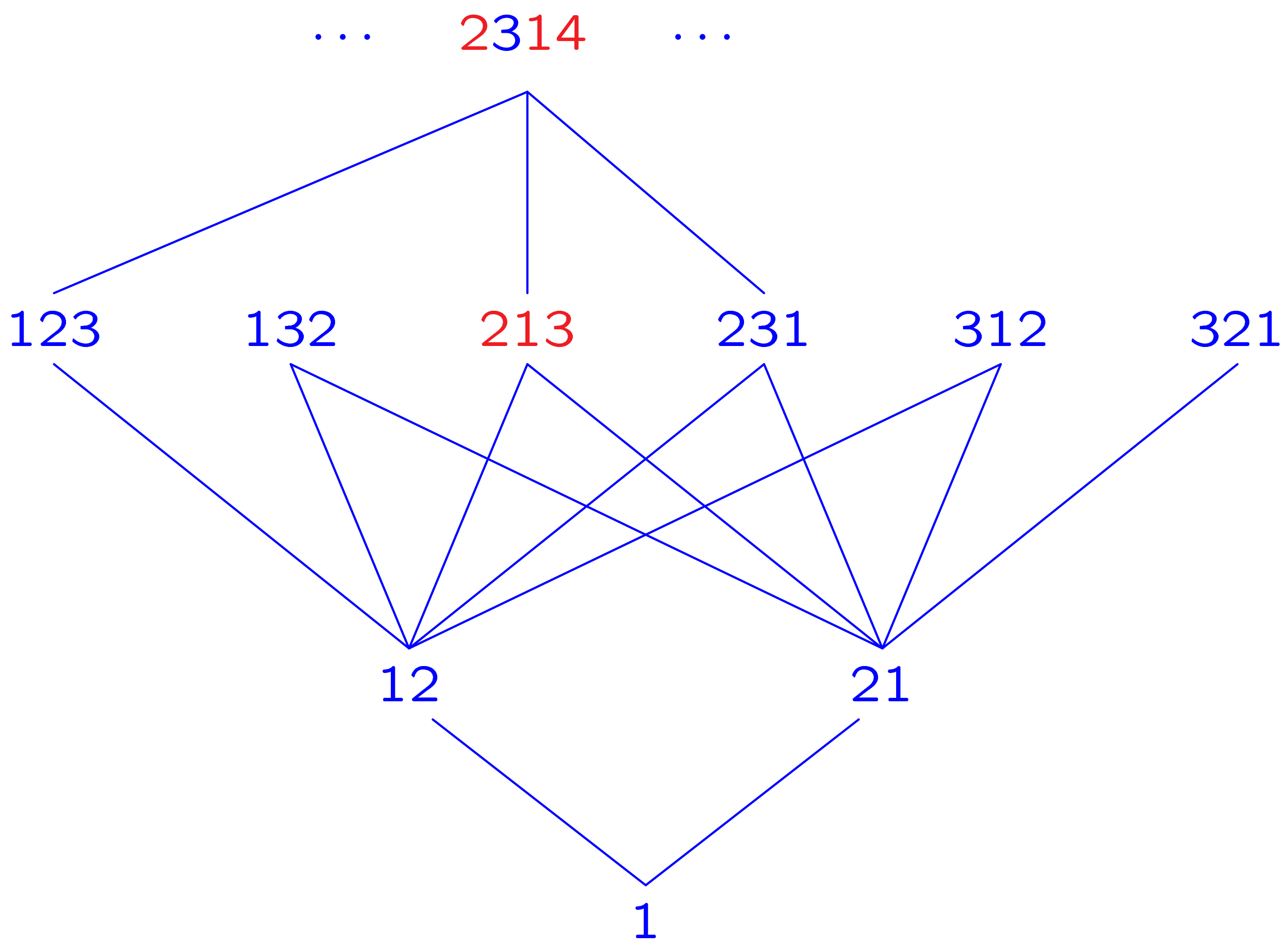
Permutation patterns and the Möbius function

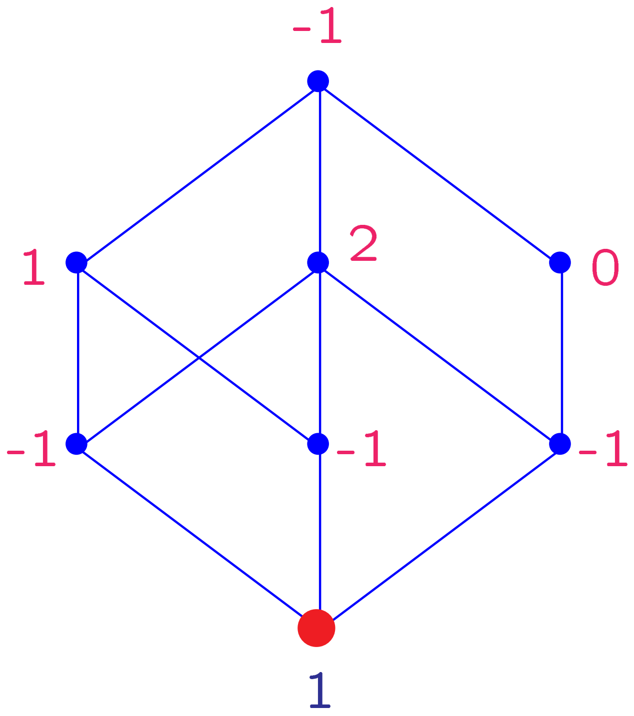
Vít Jelínek, Eva Jelínková and Einar Steingrímsson

including some joint work (in progress) with Alex Burstein



The poset of permutations w.r.t. pattern containment

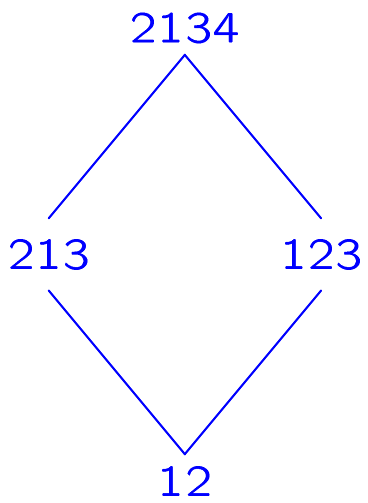
Computing the Möbius function $\mu(\bullet, y)$



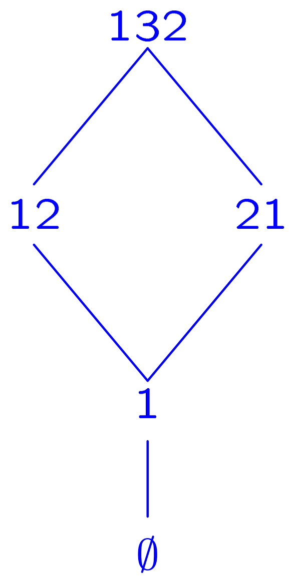
The Möbius function is defined by $\mu(x,x) = 1$ and

$$\sum_{x \leqslant t \leqslant y} \mu(x,t) = 0 \quad \text{if } x < y$$

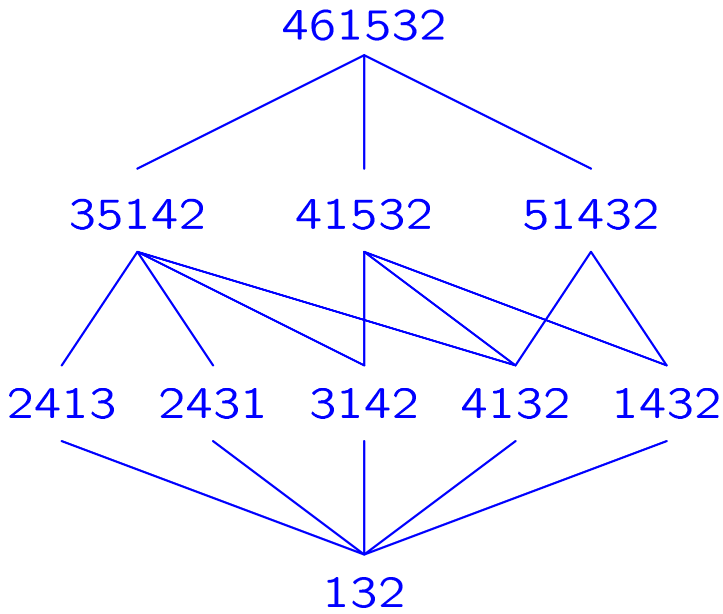
Some examples



$\mu(12, 2134) = 1$

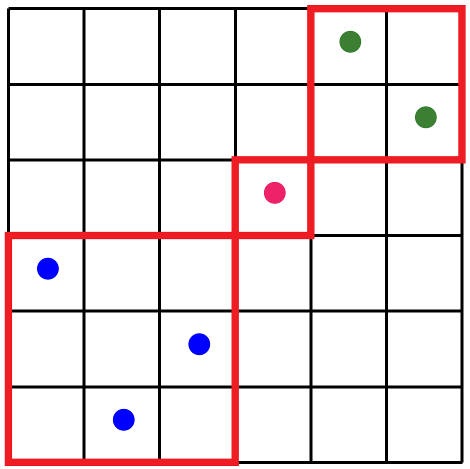


$\mu(\emptyset, 132) = 0$

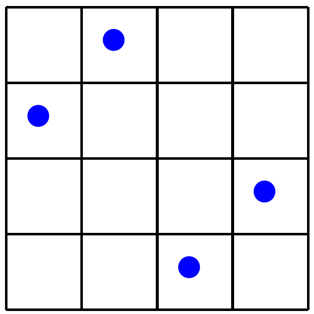


$\mu(132, 461532) = -2$

A permutation is *decomposable* if it is the *direct sum* of two or more (nonempty) permutations:



$312465 = 312 \oplus 1 \oplus 21$



Indecomposable

We write $\pi = \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_n$ only if each π_i is indecomposable

Let $\sigma = \sigma_1 \oplus \sigma_2 \oplus \cdots \oplus \sigma_m$ and $\pi = \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_n$

Let $\pi_{>i} = \pi_{i+1} \oplus \pi_{i+2} \oplus \cdots \oplus \pi_n$ etc. for $\pi_{\geqslant i}, \sigma_{>i}, \sigma_{\geqslant i}$

First Recurrence: Let $\ell \geqslant 0$ and $k \geqslant 1$ be maximal so that $\sigma_1 = \sigma_2 = \cdots = \sigma_\ell = 1$ and $\pi_1 = \pi_2 = \cdots = \pi_k = 1$. Then

$$\mu(\sigma, \pi) = \begin{cases} 0 & \text{if } \ell \leqslant k-2 \\ -\mu(\sigma_{\geqslant k}, \pi_{>k}) & \text{if } \ell = k-1 \\ \mu(\sigma_{>k}, \pi_{>k}) - \mu(\sigma_{\geqslant k}, \pi_{>k}) & \text{if } \ell \geqslant k \end{cases}$$

Example:

$\mu(132, 1237564) = 0$

$\mu(132, 126453) = -\mu(21, 4231) = -2$

$\mu(132, 13524) = \mu(21, 2413) - \mu(132, 2413) = 3 - (-1) = 4$

Main Theorem: Suppose $\pi_1 \neq 1$. Let $k \geqslant 1$ be maximal so that $\pi_1 = \pi_2 = \cdots = \pi_k$. Then

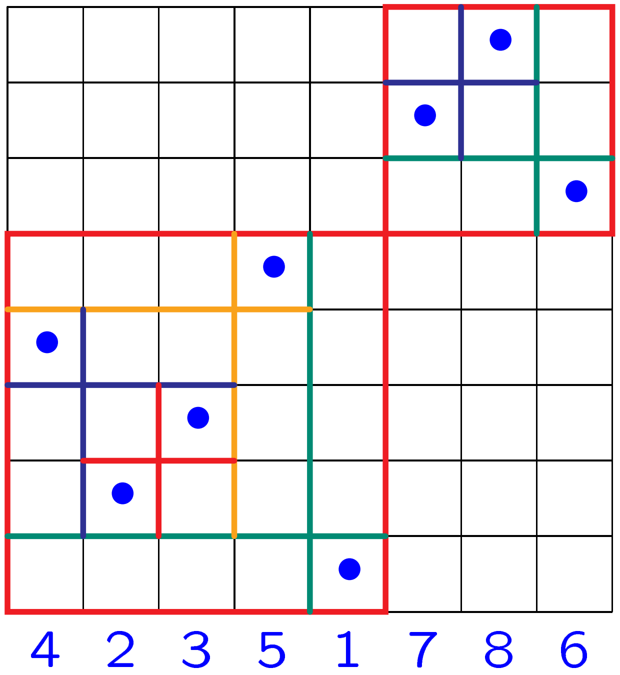
$$\mu(\sigma, \pi) = \sum_{i=1}^{m} \sum_{j=1}^{k} \mu(\sigma_{\leqslant i}, \pi_1)\mu(\sigma_{>i}, \pi_{>j})$$

Corollary: If $\sigma = a \oplus b$ and $\pi = c \oplus d$, where $c, d \neq 1, c \neq d$, then

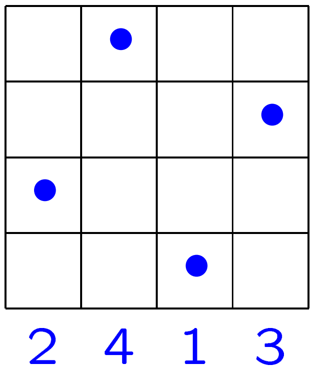
$$\mu(\sigma, \pi) = \mu(a, c) \cdot \mu(b, d)$$

Corollary: If σ is indecomposable (so $m = 1$), then

- $\mu(\sigma, \pi) = \mu(\sigma, \pi_1)$ if $\pi = \pi_1 \oplus \pi_1 \oplus \cdots \oplus \pi_1$
- $\mu(\sigma, \pi) = -\mu(\sigma, \pi_1)$ if $\pi = \pi_1 \oplus \pi_1 \oplus \cdots \oplus \pi_1 \oplus 1$ $(\pi_1 \neq 1)$
- $\mu(\sigma, \pi) = 0$ otherwise



Separable

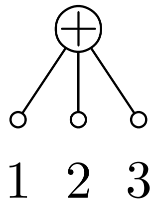


Nonseparable

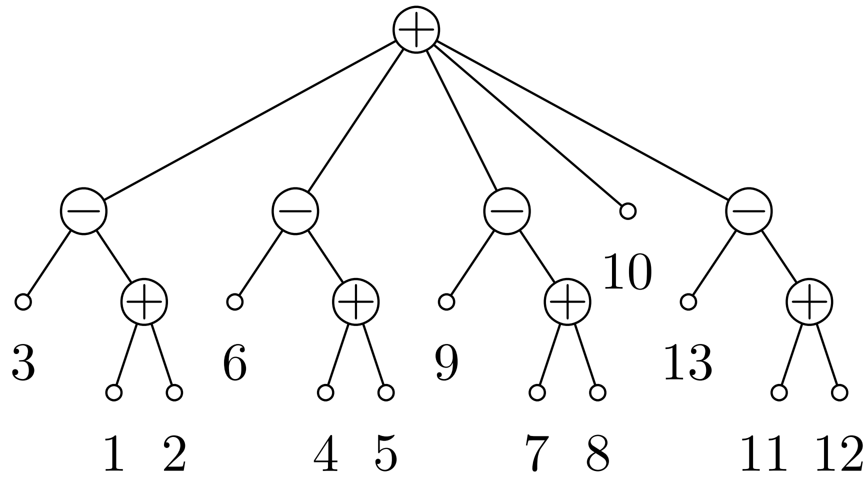
A permutation is *separable* if it can be generated from 1 by direct sums and *skew sums*.

$42351786 = 42351 \oplus 231 = (3124 \ominus 1) \oplus 231 = \cdots$

A permutation is separable if and only if it avoids the patterns 2413 and 3142.



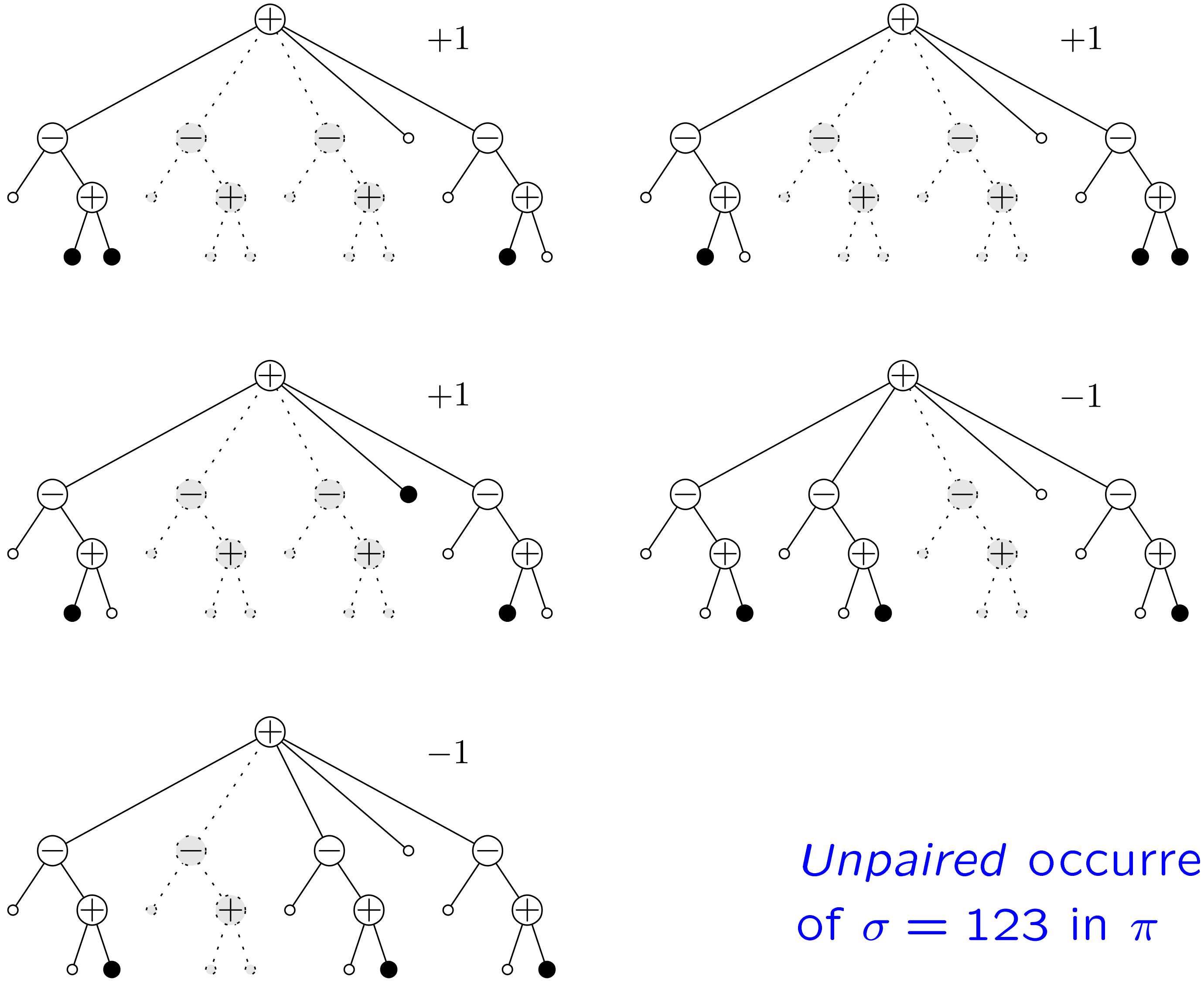
$\sigma = 123$



$\pi = 3, 1, 2, 6, 4, 5, 9, 7, 8, 10, 13, 11, 12$

The *separating trees* of σ and π

(σ and π separable)



Unpaired occurrences
of $\sigma = 123$ in π

Theorem: If σ and π are separable permutations, then

$$\mu(\sigma,\pi) = \sum_{\mathsf{X}} (-1)^{\mathsf{parity(X)}}$$

where the sum is over *unpaired* occurrences of σ in π.

Note: This computes $\mu(\sigma,\pi)$ in polynomial time, although the size of the interval $[\sigma,\pi]$ may grow exponentially.

Corollary: If π is separable then

$$|\mu(\sigma,\pi)| \leqslant \sigma(\pi)$$

where $\sigma(\pi)$ is the number of occurrences of σ in π.

In particular: If π avoids 132 then $|\mu(\sigma,\pi)| \leqslant \sigma(\pi)$

More results

- $\mu(135\ldots 2k\text{-}1\ 2k\ldots 42, 135\ldots 2n-1\ 2n\ldots 42) = \binom{n+k-1}{n-k}$
- In particular, $\mu(\sigma, \pi)$ is not bounded
- If π is separable then $\mu(1, \pi) \in \{0, 1, -1\}$
- If σ is indecomposable and $\pi = \pi_1 \oplus 1 \oplus \pi_2$ then $\mu(\sigma, \pi) = 0$

Open problems:

- When is $\mu(\sigma, \pi) = 0$?
- When is $|\mu(\sigma, \pi)| = \sigma(\pi)$?
- What is $max\{\mu(1, \pi) : |\pi| = n\}$?
- Can we relate $\mu(\sigma, \pi)$ to the homology of $[\sigma, \pi]$?
- Which intervals $[\sigma, \pi]$ are shellable?
- Conjecture: $max_{\pi \in S_n} |\mu(1, \pi)|$ is unbounded as a function of n