Sorting monoids on Coxeter groups

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arXiv:0711.1561 [math.RT] (FPSAC’06)
arXiv:0804.3781 [math.RT] (FPSAC’08)
arXiv:0912.2212 [math.CO] (FPSAC’10)
  + research in progress
Bubble (anti) sort algorithm

1234
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1234
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1243
Bubble (anti) sort algorithm

1423
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4123
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4123
Bubble sort and Coxeter groups  The cutting poset  The biHecke monoid  Combinatorics  Representation theory

Bubble (anti) sort algorithm

4132
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4312
Bubble sort and Coxeter groups  The cutting poset  The biHecke monoid  Combinatorics  Representation theory

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Underlying combinatorics: right permutahedron
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Elementary transpositions: $s_1, s_2, s_3, \ldots$
Bubble (anti) sort algorithm

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Underlying combinatorics: right permutahedron

Elementary transpositions: $s_1, s_2, s_3, \ldots$

Relations: $s_i^2 = 1, (s_1s_2)^3 = 1, (s_2s_3)^3 = 1, (s_1s_3)^2 = 1$
Coxeter groups

**Definition (Coxeter group $W$)**

**Generators:** $s_1, s_2, \ldots$ (simple reflections)

**Relations:** $s_i^2 = 1$ and $s_i s_j \cdots = s_j s_i \cdots$, for $i \neq j$

Reduced words
0-Hecke monoid
0-Hecke monoid

\[ \pi_1 \circ \pi_2 \]

\[ s_1, s_2 \]

\[ S_3 \]

\[ H_0(\mathbb{S}_3) \]
0-Hecke monoid

\[ \begin{array}{c}
\begin{array}{ccc}
321 & 321 \\
312 & 312 \\
231 & 231 \\
213 & 213 \\
132 & 132 \\
123 & 123 \\
\end{array}
\end{array} \]

\[ \begin{array}{c}
\begin{array}{ccc}
\pi_1 & \pi_2 \\
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\[
\begin{array}{ccc}
321 & & 321 \\
\downarrow s_1 & & \downarrow \pi_1 \\
231 & & 231 \\
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0-Hecke monoid

\[ S_3 \quad \xrightarrow{H_0(S_3)} \quad \bar{H}_0(S_3) \]
0-Hecke monoid

Definition (0-Hecke monoid $H_0(W)$ of a Coxeter group $W$)

Generators: $\langle \pi_1, \pi_2, \ldots \rangle$ (simple reflections)
Relations: $\pi_i^2 = \pi_i$ and braid relations

Theorem

$|H_0(W)| = |W|$ + lots of nice properties

Motivation: simple combinatorial model (bubble sort) appears in Iwahori-Hecke algebras, Schur symmetric functions, Schubert, Macdonald, Kazhdan-Lusztig polynomials, (affine) Stanley symmetric functions, mathematical physics, Schur-Weyl duality for quantum groups, representations of $GL(\mathbb{F}_q)$, ...
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Classical orders on Coxeter groups

Right order
Classical orders on Coxeter groups

Right order

Prefix
Classical orders on Coxeter groups

Left order

Suffix

Right order

Prefix
Classical orders on Coxeter groups

Left order

Left-Right order

Right order

Suffix

Factor

Prefix
Classical orders on Coxeter groups

Left order

Left-Right order

Right order

Bruhat order

Suffix

Factor

Prefix

Subword
# Blocks of permutations

## Definition (Block of a permutation $w$)

- **Type $A$:** sub-permutation matrix
- **Type free:** $J, K$ such that $W_J w = w W_K$

## Example

Let $w := 36475812$

![Diagram of a permutation matrix](image)

- Simple permutation: cf. [Albert, Atkinson 05] + dim 2 posets
- $\{\text{blocks of } w\}$: sub-lattice of the Boolean lattice
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The cutting poset

Definition (HST09: Cutting poset $(W, \trianglelefteq)$)

$u \trianglelefteq w$ if $u = w^J$ with $J$ block

Theorem

- Intervals are lattices
- Möbius function: inclusion-exclusion along minimal blocks
- Meet-semi lattice?
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The Big Picture

$$\text{NDPF}(\text{Bruhat}(W)) \quad \text{End}(<_L(W)) \quad \text{End}(\text{BooleanLattice})$$

$$M_1 \quad \langle \pi_1, \pi_2, \ldots, \bar{\pi}_1, \bar{\pi}_2, \ldots \rangle \quad \langle \pi_1, \pi_2, \ldots, s_1, s_2, \ldots \rangle$$

$$H_\zeta(\tilde{W}) \quad H_{-1}(\tilde{W}) \quad H_0(\tilde{W}) \quad H_0(\bar{W}) \quad H_0(\bar{W})$$

$$\langle \bar{\pi}_0 \pi_1, \pi_2, \ldots \rangle \quad \langle \bar{\pi}_0 \bar{\pi}_1, \pi_2, \ldots \rangle \quad \langle \pi_1, \pi_2, \ldots, s_1, s_2, \ldots \rangle \quad \langle \pi_0 \pi_1, \pi_2, \ldots \rangle$$

$$\tilde{W} \quad H^W \quad W \quad H_q(\tilde{W}) \quad H_q(W)$$

$$\langle s_0 s_1, s_2, \ldots \rangle \quad Q[\pi_1, \pi_2, \ldots, s_1, s_2, \ldots] \quad Q[\pi_1, \pi_2, \ldots, \bar{\pi}_1, \bar{\pi}_2, \ldots]$$

$$\langle \pi_1, \pi_2, \ldots, \bar{\pi}_1, \bar{\pi}_2, \ldots \rangle$$

$$H_\zeta(S_n) \otimes \wedge \quad TL_n \quad \text{NDPF}_n \quad \text{NDPF}_B$$

$$\text{NDPF}_n \quad \text{NDF}_n$$

$$S_n \otimes \wedge \quad H_q(S_n) \otimes \wedge$$
The biHecke monoid

<table>
<thead>
<tr>
<th>Question</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Size of</strong> $M(W) = \langle \pi_1, \pi_2, \ldots, \overline{\pi}_1, \overline{\pi}_2, \ldots \rangle$</td>
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- How to attack such a problem?
- Generators and relations?
- Representation theory?

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<th>Theorem (HST08)</th>
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- Why do we care?

\[
|M(W)| = \sum_{w \in W} \dim S_w \cdot \dim P_w
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**The biHecke monoid**

### Question

*Size of $M(W) = \langle \pi_1, \pi_2, \ldots, \bar{\pi}_1, \bar{\pi}_2, \ldots \rangle$*

$|M(S_n)| = 1, 3, 23, 477, 31103, ?$

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$M(W)$ admits $|W|$ simple / indecomposable projective modules

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**Key combinatorial lemma**

**Lemma**

For $f \in M(W)$ and $w \in W$: $(s_i w) . f = w . f$ or $s_i (w . f)$

**Proof.**

Exchange property / associativity
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Corollary

- Preservation of left order: $u \leq_L v \implies u.f \leq_L v.f$
- Preservation of Bruhat order: $u \leq_B v \implies u.f \leq_B v.f$
- $M(W)$ is aperiodic
- $f$ in $M(W)$ is determined by its fibers and $f(1)$
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The image set of an idempotent is an interval in left order
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Green relations
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Theorem

- **Regular \( J \)-classes are indexed by \( W \)
- \( J \)-order on regular classes: left-right order on \( W \)
- \( R \)-classes: intervals in right order on \( W \)
- \( R \)-order on regular \( R \)-classes: \( \approx \) right order on \( W \)
- \( L \)-order on regular \( L \)-classes: \( \approx \) left order on \( W \)

Problems

- \( L, R, J \)-order between non regular classes?
- \( L \)-classes?
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\[ M(W) \text{ admits } |W| \text{ simple modules / indecomposable projective modules} \]

**Problem**

*Dimension of simple and indecomposable projective modules?*
### Corollary

$M(W)$ admits $|W|$ simple modules / indecomposable projective modules

### Problem

*Dimension of simple and indecomposable projective modules?*
The “Borel” submonoid $M_1$

**Definition**

Submonoid $M_1 := \{ f \in M, f(1) = 1 \}$

**Properties**

- Weakly increasing and contracting on Bruhat $\implies J$-trivial
- Idempotents: $(e_w)_{w \in W}$
- Generated by $e_w$ for $w$ grassmanian, e.g. atom for $(W, \vee_L)$
- $|W|$ simple modules of dimension 1
- Semi-simple quotient: monoid algebra of $(W, \vee_L)$
- Conjugacy order among idempotents: $<_L$
- $\dim P_w = |\{ f \in M_1, f(w) \leq_L w \}| = ?$

**Problem**

*Inducing these results to $M$?*
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\( \mathcal{H} W^{(w)} := \mathbb{Q}[\pi_1, \pi_2, \ldots, \overline{\pi}_1, \overline{\pi}_2, \ldots] \) acting on \( \mathbb{Q}.[1, w]_R \)

- Blocks: \( J = \{\}, \{1, 2\}, \{3\}, \{1, 2, 3\} \implies \) Submodules \( P_J \)
- \( \mathcal{H} W^{(w)} \): max. algebra stabilizing all \( P_J \) \implies Repr. theory
- \( \mathcal{H} W^{(w)} \) quotient of \( \mathbb{Q}[M(W)] \); top: simple module \( S_w \) of \( M \)
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- Generating series calculation?
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**R-classes and translation algebras**

![Diagram of R-classes and translation algebras](image)

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- **\( H_W(w) \):** max. algebra stabilizing all \( P_J \) \quad \Rightarrow \quad \text{Repr. theory}
- **\( H_W(w) \)** quotient of \( \mathbb{Q}[M(W)] \); top: simple module \( S_w \) of \( M \)
- **Dimension:** inclusion-exclusion along the cutting poset
- **Generating series calculation?**
Definition (Translation algebra)

\( \mathcal{H}W^{(w)} := \mathbb{Q}[\pi_1, \pi_2, \ldots, \bar{\pi}_1, \bar{\pi}_2, \ldots] \) acting on \( \mathbb{Q}.[1, w]_R \)

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Work in progress

- $\mathcal{L}$-classes? Projective modules? Cartan Matrix?

- Generalization to $R$-trivial and aperiodic monoids (collaboration with Denton and Berg, Bergeron, Saliola)

- Fast implementation in Sage (interface with Semigroupe, ...)

Sage-Combinat meeting tonight

Sage’s mission:

“To create a viable high-quality and open-source alternative to Maple™, Mathematica™, Magma™, and MATLAB™”

... 

“and to foster a friendly community of users and developers”

Tonight, Thurston Hall, Room 236

- 7pm-8pm: Introduction to Sage and Sage-Combinat
- 8pm-10pm: Help on installation & getting started
  Bring your laptop!
- Design discussions