Schubert complexes and degeneracy loci

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Introduction

X is a smooth quasi-projective variety and $\varphi\colon E \to F$ is a "generic" map of vector bundles over X. (rank $E = e$, rank $F = f$) We want to understand various degeneracy loci $D(\varphi)$ associated with φ:

- Find combinatorial formulas for $[D(\varphi)]$ in cohomology (Chow) ring $\mathrm{A}(X)$.
- Find combinatorial formulas for $[\mathcal{O}_{D(\varphi)}]$ in K-theory $\mathrm{K}(X)$.

This has been done in many situations, we focus on the first problem.

Thom–Porteous formula

$D_r(\varphi) = \{x \in X \mid \operatorname{rank}(\varphi(x)\colon E(x) \to F(x) \le r\}$ for some fixed r.
Given a partition λ, let $s_\lambda(x)$ be the corresponding Schur polynomial. Define the **super Schur polynomial** by

$$s_\lambda(x;y) = \sum_{\mu \subseteq \lambda} s_\mu(x) s_{\lambda'/\mu'}(-y).$$

For $\lambda = (f-r) \times (e-r)$, x and y are the Chern roots of F and E, respectively, then $[D(\varphi)] = s_\lambda(x;y)$. [This function is symmetric in x and y separately, so order is not important.]

Fulton's formula [F1]

Suppose we are given subbundles $E_1 \subset \cdots \subset E_{e-1} \subset E$ and quotient bundles $F \twoheadrightarrow F_{f-1} \twoheadrightarrow \cdots \twoheadrightarrow F_1$ (indices denote rank). Let $w \in \Sigma_n$ be a permutation. Set $r_w(p,q) = \{i \le p \mid w(i) \le q\}$ and

$$D_w(\varphi) = \{x \in X \mid \operatorname{rank}(\varphi(x)\colon E_p(x) \to F_q(x)) \le r_w(p,q)\}.$$

Let $\mathfrak{S}_w(x)$ be the Schubert polynomial. Define the **double Schubert polynomial**

$$\mathfrak{S}_w(x;y) = \sum_u \mathfrak{S}_u(x) \mathfrak{S}_{uw^{-1}}(-y)$$

(sum is over u whose inversion set is contained in inversion set of w). Then setting $-x_i$ and $-y_j$ to be the Chern class of E_i/E_{i-1} and $\ker(F_j \twoheadrightarrow F_{j-1})$, respectively, we have $[D(\varphi)] = \mathfrak{S}_w(x;y)$. [This function is NOT symmetric in x nor y in general, so order is important.]

Approximation by Cohen–Macaulay modules

- $\mathrm{K}(X) \otimes \mathbf{Q} \cong \mathrm{A}(X) \otimes \mathbf{Q}$ via Chern character, but this map can be hard to use. Also, the naïve way to get a formula for $[\mathcal{O}_{D(\varphi)}]$ in K-theory is to write down a locally free resolution, but this is very hard to do in general.
- Workaround: Let $\mathrm{F}_i\mathrm{K}(X)$ be spanned by coherent sheaves whose support has dimension at most i. This filtration respects ring structure and the associated graded $\mathrm{gr}\,\mathrm{K}(X)$ is isomorphic to cohomology in an "easier way."
- Suggestion: instead of resolving $\mathcal{O}_{D(\varphi)}$, we find some coherent sheaf $\mathcal{M}$ that is equal to $\mathcal{O}_{D(\varphi)}$ plus lower order terms and which has a resolution that is related to the formulas we already have. More precisely, these degeneracy loci are torus-invariant for an appropriate torus and we require:
  - The torus-equivariant Euler characteristic of the resolution of $\mathcal{M}$ is a super Schur polynomial or double Schubert polynomial
  - There are no cancellations in the general case in the above Euler characteristic

Approximating $D_r(\varphi)$ with Schur complexes

- **Schur functors $\mathbf{S}_\lambda$** take in a vector space V and spit out a representation of the Lie algebra $\mathfrak{gl}(V)$. Their character is the Schur polynomial $s_\lambda(x)$.
- **Super Schur functors** take in a super vector space $V_0 \oplus V_1$ and spit out a representation of the Lie superalgebra $\mathfrak{gl}(V_0|V_1)$. Their (super)character is the super Schur polynomial $s_\lambda(x;y)$. If we give a map $V_1 \to V_0$, then the representation has the structure of a chain complex (**Schur complex**). These complexes were introduced and their properties below were proved in [ABW].
- Also works for vector bundles. Taking $\lambda = (f-r) \times (e-r)$ and $V_0 = F$ and $V_1 = E$, the chain complex will be acyclic, and it resolves a Cohen–Macaulay sheaf that approximates $D_r(\varphi)$. The genericity condition means that the ideal generated by $(r+1) \times (r+1)$ minors of φ has depth $(f-r)(e-r)$.
- **Generating functions:** Let T be a filling of the Young diagram of λ with the entries $x_1 < x_2 < \cdots < -y_1 < -y_2 < \cdots$ subject to the rules:
  - Entries in a row (left to right) or column (top to bottom) are increasing.
  - No repeats of x_i in any column and no repeats of $-y_j$ in any row.

This is a **super semistandard Young tableaux** ($\mathrm{S^3YT}$). Let $m(T)$ be the product of the entries, then $s_\lambda(x;y) = \sum_T m(T)$. Also, $\mathbf{S}_\lambda(V_0 \oplus V_1)$ has a basis indexed by $\mathrm{S^3YT}$.

Approximating $D_w(\varphi)$ with Schubert complexes

- **Schubert functors** [KP] $\mathcal{S}_w$ take in a flag of vector spaces $V^1 \subset V^2 \subset \cdots \subset V$ and spit out a representation of the Borel subalgebra $\mathfrak{b}(V^\bullet)$ of matrices that preserve this flag.
- We introduce **double Schubert functors** that take in a flag of super vector spaces $V^{-n} \subset \cdots \subset V^{-1} = V_1 \subset V^1 \subset \cdots \subset V^n = V_0 \oplus V_1$ and spit out a representation of the Borel subsuperalgebra $\mathfrak{b}(V^\bullet)$ of matrices that preserve this flag. Here V_1 is the subspace of odd elements. If we give a map $V_0 \to V_1$, then the representation has the structure of a chain complex (**Schubert complex**).
- Also works for vector bundles. **Our main result:** Taking $F = V_1$ and $E = V_0$, $V^i = F \oplus E_i$, $V^{-j} = \ker(F \twoheadrightarrow F^{j-1})$, the complex $\mathcal{S}_w$ is acyclic and it resolves a Cohen–Macaulay sheaf that approximates $D_w(\varphi)$. The genericity condition in the introduction can be made precise by requiring that an ideal generated by certain minors of φ has depth $\ell(w)$.
- **Generating functions:** Let $D(w) = \{(i, w(j)) \mid i < j, w(i) > w(j)\} \subset \mathbf{Z}_{>0} \times \mathbf{Z}_{>0}$ with matrix conventions for indexing pairs. Let T be a filling of $D(w)$ with the entries $\cdots < -y_2 < -y_1 < x_1 < x_2 < \cdots$ subject to the rules:
  - No entry in the ith row is strictly bigger than x_i and no entry in the jth column is strictly smaller than $-y_j$.
  - No repeats of x_i in any column and no repeats of $-y_j$ in any row.
  - Given any $(i,j) \in D(w)$, let $H(i,j)$ be the set of boxes of $D(w)$ in the same row to the right of (i,j) or in the same column below (i,j) (including (i,j) itself). Take all of the entries in $H(i,j)$ and rearrange them in weakly increasing order starting from the top right end to the bottom left end. The entry in (i,j) must stay the same.

This is a **balanced super labeling** (SBL). Let $m(T)$ be the product of the entries, then $\mathfrak{S}_w(x;y) = \sum_T m(T)$. If we don't allow the $-y_j$ to appear, we get the single Schubert polynomials, and this generating function was given by [FGRS]. Also $\mathcal{S}_w(V^\bullet)$ has a basis indexed by the SBL.

Remarks on the proof

To show acyclicity of $\mathcal{S}_w$, we adapted a filtration on Schubert functors introduced in [KP] to the Schubert complex case. The quotients of this filtration are Schubert complexes for "smaller" permutations so one can use induction arguments via long exact sequences on homology. The main difficulty to applying this is that one of the quotients has a grading shift so that the resulting long exact sequence does not immediately imply acyclicity of $\mathcal{S}_w$, so many tricks were employed:

- **Geometry:** We first work on the flag variety, and crucially use that the cohomology classes of Schubert varieties are given by Schubert polynomials and that Schubert varieties are normal.
- **Combinatorics:** It was necessary to explicitly show that certain minors of φ annihilate the cokernel of $\mathcal{S}_w(\varphi)$. To do so, we needed the fact that SBLs span $\mathcal{S}_w(\varphi)$.
- **Algebra:** The Auslander–Buchsbaum formula was indispensable for obtaining certain inequalities that forced the homology of $\mathcal{S}_w(\varphi)$ to be 0. The general case is deduced from the flag variety case by using the Buchsbaum–Eisenbud acyclicity criterion.

Further directions

- **Schubert polynomials in other types:** One can interpret Fulton's degeneracy loci as a type A instance. One could endow the vector bundles involved with nondegenerate bilinear forms and consider "type BCD" instances. These degeneracy loci have also been considered [F2] and involve analogues of double Schubert polynomials. Do these give rise to complexes?
- **Quiver degeneracy loci:** Let Q be a directed graph. On each vertex put a vector bundle and on each arrow put a map between those vector bundles. We can impose rank conditions on various compositions of these arrows. These have been studied intensively and combinatorial formulas in the case of the graph $\bullet \to \bullet \to \cdots \to \bullet$ are given in [KMS]. Are there Cohen–Macaulay approximations lurking in the background?
- **Resolving $\mathcal{O}_{D(\varphi)}$:** Lascoux [Las] described the resolutions of $\mathcal{O}_{D_r(\varphi)}$ explicitly in the characteristic 0 case. What can we say about the resolutions of $\mathcal{O}_{D_w(\varphi)}$? The terms of these resolutions can look different in positive characteristic (even though the Euler characteristic is unchanged), so it is unlikely that there is a "combinatorial" construction for them.
- **Geometric considerations:** The Schur complex can be constructed geometrically as follows. Let E and F be vector spaces and Y the subvariety of $\mathrm{Hom}(E,F)$ consisting of maps of rank $\le r$. Then Y has a desingularization given by a vector bundle Z over a Grassmannian. There is a "twisted Koszul complex" on Z whose (derived) pushforward to $\mathrm{Hom}(E,F)$ is the Schur complex. Does such a construction exist for Schubert complexes?

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