Cyclic sieving for longest reduced words in the hyperoctahedral group

Luis Serrano
Université du Québec à Montréal

Joint work with Kyle Petersen (DePaul University)

FPSAC, San Francisco, California
August, 2010

Longest words in the hyperoctahedral group

Hyperoctahedral group: B_n

Generators: $s_0, s_1, \ldots, s_{n-1}$

Relations: $$\left\{ \begin{array}{rcl} s_i^2 & = & 1 \\ s_i\, s_j & = & s_j\, s_i \;\text{ for }\; |i-j| \geq 2 \\ s_i\, s_{i+1}\, s_i & = & s_{i+1}\, s_i\, s_{i+1} \;\text{ for }\; i \geq 1 \\ s_0\, s_1\, s_0\, s_1 & = & s_1\, s_0\, s_1\, s_0. \end{array} \right\}$$

Longest element: w_0, of length $\ell(w_0) = n^2$

$R(w_0) = \{\text{reduced words for } w_0\}$

Cyclic rotation: $a_1 a_2 \cdots a_{n^2} \stackrel{\omega}{\mapsto} a_2 \cdots a_{n^2} a_1$

Q: What are the sizes of the orbits with respect to this action?

Example: B_3

Orbit of size 9:

$$\xrightarrow{\omega} \mathbf{0}10212012 \xrightarrow{\omega} 10212012\mathbf{0} \xrightarrow{\omega} 021201201 \xrightarrow{\omega} 212012010$$
$$\xrightarrow{\omega} 120120102 \xrightarrow{\omega} 201201021 \xrightarrow{\omega} 012010212 \xrightarrow{\omega} 120102120$$
$$\xrightarrow{\omega} 201021201 \xrightarrow{\omega}$$

Orbit of size 3:

$$\xrightarrow{\omega} \mathbf{0}12012012 \xrightarrow{\omega} 12012012\mathbf{0} \xrightarrow{\omega} 201201201 \xrightarrow{\omega}$$

42 words in $R(w_0)$:

- 42 words fixed by 0 rotations,
- 6 words fixed by 3 rotations (example: 012012012),
- 6 words fixed by 6 rotations,
- 0 words fixed by any other number of rotations (mod 9),

Square Young tableaux

$SYT(n^n) = \{\text{Standard Young tableaux of shape } n^n\}$

Promotion:

$$\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline 7&8&9\\\hline\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline &2&4\\\hline 3&5&6\\\hline 7&8&9\\\hline\end{array} \xrightarrow{\text{jdt}} \begin{array}{|c|c|c|}\hline 2&4&6\\\hline 3&5&9\\\hline 7&8&\\\hline\end{array} \xrightarrow{-1} \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&8\\\hline 6&7&\\\hline\end{array} \longrightarrow \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&8\\\hline 6&7&9\\\hline\end{array}$$

So

$$\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5&6\\\hline 7&8&9\\\hline\end{array} \xrightarrow{\omega} \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4&8\\\hline 6&7&9\\\hline\end{array}$$

Example: $SYT(3^3)$

Promotion orbit of size 3

$$\xrightarrow{\omega} \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline \end{array} \xrightarrow{\omega} \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 & 8 \\ \hline 6 & 7 & 9 \\ \hline \end{array} \xrightarrow{\omega} \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 6 & 7 \\ \hline 5 & 8 & 9 \\ \hline \end{array} \xrightarrow{\omega}$$

42 tableaux in $SYT(3^3)$:

- 42 tableaux fixed by 0 promotions,
- 6 tableaux fixed by 3 promotions,
- 6 tableaux fixed by 6 promotions,
- 0 tableaux fixed by any other number of promotions (mod 9),

Cyclic sieving phenomenon (CSP)

X a set.

$C = \langle \omega \rangle$ a finite cyclic group acting on X.

$X(q) \in \mathbb{Z}(q)$ a polynomial in q.

The triple $(X, C, X(q))$ exhibits CSP if for all $d \geq 0$, the number of elements fixed by ω^d is $X(\zeta^d)$, where ζ is a primitive root of unity of order $|C|$.

Cyclic sieving in $SYT(n^n)$

Theorem (Rhoades)

The following triple exhibits CSP:

$X = SYT(n^n)$

$\omega = \textit{promotion}$

$$X(q) = \frac{[n^2]!_q}{\prod_{(i,j)\in(n^n)} [h_{i,j}]_q}$$ *(the q-hook polynomial)*

In $SYT(3^3)$, for $\zeta = e^{\frac{2i\pi}{9}}$

- $X(\zeta^0) = X(1) = 42$,
- $X(\zeta^3) = 6$,
- $X(\zeta^6) = 6$,
- $X(\zeta^i) = 0$ for $i \neq 0, 3$, or $6 \pmod 9$.

Main theorem

Major index: sum of the positions of the descents

$$w = 0\mathbf{1}0\mathbf{2}1\mathbf{2}012$$

$$\mathrm{maj}(w) = 2 + 4 + 6 = 12$$

Theorem (Petersen - Serrano)

The following triple exhibits CSP:

$X = R(w_0)$ *(the set of reduced words for w_0)*

$\omega =$ *cyclic rotation*

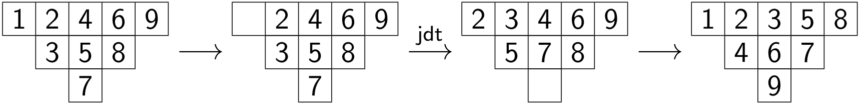
$$X(q) = q^{-n\binom{n}{2}} \sum_{w \in R(w_0)} q^{\mathrm{maj}(w)}$$

Sketch of proof of the main theorem

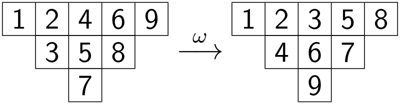
- Bijection H between $R(w_0)$ and $SYT(n^n)$.
- H behaves well with respect to CSP.
  - Cyclic rotation corresponds to promotion.
  - Polynomials are the same.
- CSP follows from Rhoades's theorem.
- Note: The bijection goes through an intermediate object: *double staircases*.

Shifted double staircases

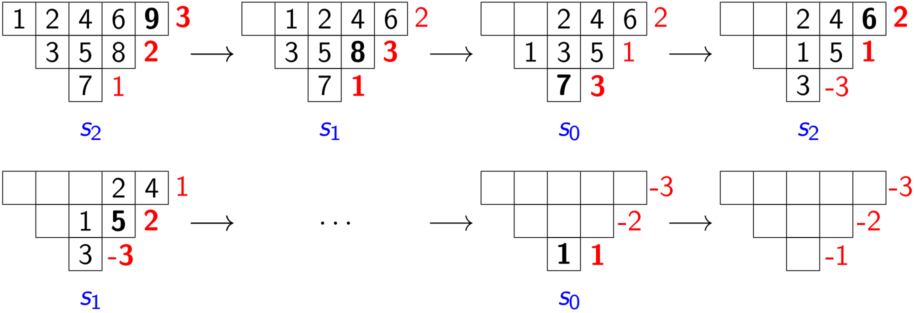
$SYT'(2n-1, 2n-3, \ldots, 1) = \{\text{shifted double staircases}\}$
Promotion:



So



Bijection between longest reduced words and shifted double staircases (Haiman)



Define $H_1\left(\begin{array}{ccccc} 1 & 2 & 4 & 6 & 9 \\ & 3 & 5 & 8 & \\ & & 7 & & \end{array}\right) = s_0 s_1 s_0 s_2 s_1 s_2 s_0 s_1 s_2$. (or 010212012)

Bijection between shifted double staircases and square Young tableaux

Theorem (Haiman)

The sets $SYT(n^n)$ *and* $SYT'(2n-1, 2n-3, \dots, 1)$ *are in bijection.*

Example

$$H_2\left(\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline \end{array}\right) = \begin{array}{ccccc} 1 & 2 & 4 & 6 & 9 \\ & 3 & 5 & 8 & \\ & & 7 & & \end{array} .$$

Bijection:

$$\begin{array}{ccccc} \square & \square & 1 & 2 & 4 \\ & \bullet & 3 & 5 & 6 \\ & & 7 & 8 & 9 \end{array} \xrightarrow{\text{jdt}} \begin{array}{ccccc} \square & \bullet & 1 & 2 & 4 \\ & 3 & 5 & 6 & 9 \\ & & 7 & 8 & \end{array} \xrightarrow{\text{jdt}} \begin{array}{ccccc} \bullet & 1 & 2 & 4 & 9 \\ & 3 & 5 & 6 & \\ & & 7 & 8 & \end{array} \xrightarrow{\text{jdt}} \begin{array}{ccccc} 1 & 2 & 4 & 6 & 9 \\ & 3 & 5 & 8 & \\ & & 7 & & \end{array}$$

Proof of the main theorem

$SYT \xrightarrow{H_2} SYT' \xrightarrow{H_1} R(w_0)$

1	2	4
3	5	6
7	8	9

$\mapsto$

1	2	4	6	9
	3	5	8	
		7		

$\mapsto$ 010212012

Lemma (Petersen - Serrano)

$H_1 \circ H_2$ takes promotion in $SYT(n^n)$ to cyclic rotation in $R(w_0)$.

Lemma (Petersen - Serrano)

The q-hook polynomial in for (n^n) is $q^{-n\binom{n}{2}}$ times the major index generating function in $R(w_0)$.

$$\frac{[n^2!]_q}{\prod_{(i,j)\in(n^n)}[h_{i,j}]_q} = q^{-n\binom{n}{2}} \sum_{w\in R(w_0)} q^{\mathrm{maj}(w)}.$$

Questions

- Is there an explicit CSP for the set of shifted double staircases?
- Are there similar CSP results for longest words in other Coxeter groups?
- Rhoades's Theorem is the type A version of a more general conjecture regarding cominuscule posets. This has been proved for all finite types except B_n and checked [Dilks, Petersen, Stembridge, Yong] for B_n with $n \leq 6$.

References

T. Kyle Petersen and Luis Serrano, *Cyclic sieving for longest reduced words in the hyperoctahedral group*. arXiv: 0905.2650.