Combinatorial interpretations for γ-vectors

T. Kyle Petersen
DePaul University

joint with Eran Nevo (Cornell), arXiv:0909.0694

San Francisco, January, 2010

Combinatorial interpretations for γ-vectors

The f- and h-vectors

Let Δ be an $(n-1)$-dimensional simplicial complex, $f_k(\Delta) =$ number of faces of dimension $k-1$

$$f(\Delta; t) := \sum_{k=0}^{n} f_k(\Delta) t^k$$

$(f_0, f_1, \ldots, f_n)$ is the *f-vector*

$$h(\Delta; t) := (1-t)^n f(\Delta; t/(1-t)) = \sum_{k=0}^{n} h_k(\Delta) t^k$$

$(h_0, h_1, \ldots, h_n)$ is the *h-vector*

The f- and h-vectors

Let Δ be an $(n-1)$-dimensional simplicial complex, $f_k(\Delta) =$ number of faces of dimension $k-1$

$$f(\Delta; t) := \sum_{k=0}^{n} f_k(\Delta) t^k$$

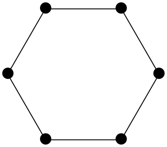
$(f_0, f_1, \ldots, f_n)$ is the *f-vector*

$$h(\Delta; t) := (1-t)^n f(\Delta; t/(1-t)) = \sum_{k=0}^{n} h_k(\Delta) t^k$$

$(h_0, h_1, \ldots, h_n)$ is the *h-vector*

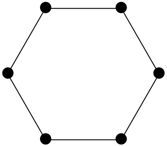
f-vectors are characterized by the *Kruskal-Katona inequalities*

The f- and h-vectors

Δ : 

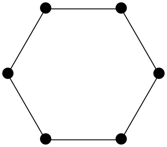
The f- and h-vectors

Δ :



- $f_0 = 1$

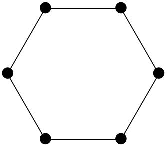
The f- and h-vectors

Δ : 

- $f_0 = 1$
- $f_1 = 6$

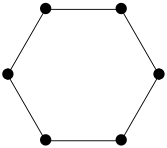
The f- and h-vectors

Δ :



- $f_0 = 1$
- $f_1 = 6$
- $f_2 = 6$

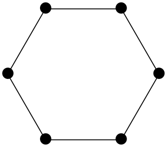
The f- and h-vectors

$\Delta :$ 

- $f_0 = 1$
- $f_1 = 6$
- $f_2 = 6$

$$f(\Delta; t) = 1 + 6t + 6t^2$$

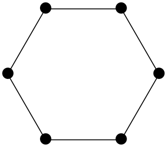
The f- and h-vectors

Δ : 

- $f_0 = 1$
- $f_1 = 6$
- $f_2 = 6$

$$\begin{aligned} f(\Delta; t) &= 1 + 6t + 6t^2 \\ &= (1 + 2t + t^2) \end{aligned}$$

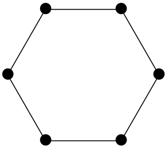
The f- and h-vectors

Δ : 

- $f_0 = 1$
- $f_1 = 6$
- $f_2 = 6$

$$\begin{aligned} f(\Delta; t) &= 1 + 6t + 6t^2 \\ &= (1 + 2t + t^2) \\ &\qquad + 4t(1+t) \end{aligned}$$

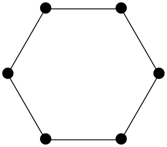
The f- and h-vectors

Δ : 

- $f_0 = 1$
- $f_1 = 6$
- $f_2 = 6$

$$\begin{aligned} f(\Delta; t) &= 1 + 6t + 6t^2 \\ &= (1 + 2t + t^2) \\ &\quad + 4t(1 + t) \\ &\quad + t^2(1) \end{aligned}$$

The f- and h-vectors

Δ : 

- $f_0 = 1$
- $f_1 = 6$
- $f_2 = 6$

$$
\begin{aligned}
f(\Delta; t) &= 1 + 6t + 6t^2 \\
&= (1 + 2t + t^2) \\
&\quad + 4t(1 + t) \\
&\quad + t^2(1) \\
h(\Delta; t) &= 1 + 4t + t^2
\end{aligned}
$$

The γ-vector

If $h(t) = \sum_{i=0}^{n} h_i t^i$ is symmetric, then there exist γ_i such that

$$h(t) = \sum_{i=0}^{n/2} \gamma_i t^i (1+t)^{n-2i},$$

The γ-vector

If $h(t) = \sum_{i=0}^{n} h_i t^i$ is symmetric, then there exist γ_i such that

$$h(t) = \sum_{i=0}^{n/2} \gamma_i t^i (1+t)^{n-2i},$$

e.g.,

$$\frac{1 + 3t + 7t^2 + 3t^3 + t^4}{}$$

The γ-vector

If $h(t) = \sum_{i=0}^{n} h_i t^i$ is symmetric, then there exist γ_i such that

$$h(t) = \sum_{i=0}^{n/2} \gamma_i t^i (1+t)^{n-2i},$$

e.g.,

$$\frac{1 + 3t + 7t^2 + 3t^3 + t^4}{1 + 4t + 6t^2 + 4t^3 + t^4}$$

The γ-vector

If $h(t) = \sum_{i=0}^{n} h_i t^i$ is symmetric, then there exist γ_i such that

$$h(t) = \sum_{i=0}^{n/2} \gamma_i t^i (1+t)^{n-2i},$$

e.g.,

$$\frac{1 + 3t + 7t^2 + 3t^3 + t^4}{\begin{array}{c} 1 + 4t + 6t^2 + 4t^3 + t^4 \\ -t(1 + 2t + t^2) \end{array}}$$

The γ-vector

If $h(t) = \sum_{i=0}^{n} h_i t^i$ is symmetric, then there exist γ_i such that

$$h(t) = \sum_{i=0}^{n/2} \gamma_i t^i (1+t)^{n-2i},$$

e.g.,

$$\begin{array}{c} 1 + 3t + 7t^2 + 3t^3 + t^4 \\ \hline 1 + 4t + 6t^2 + 4t^3 + t^4 \\ -t(1 + 2t + t^2) \\ +3t^2 \end{array}$$

The γ-vector

If $h(t) = \sum_{i=0}^{n} h_i t^i$ is symmetric, then there exist γ_i such that

$$h(t) = \sum_{i=0}^{n/2} \gamma_i t^i (1+t)^{n-2i},$$

e.g.,

$$\frac{1 + 3t + 7t^2 + 3t^3 + t^4}{\begin{array}{c} 1 + 4t + 6t^2 + 4t^3 + t^4 \\ -t(1 + 2t + t^2) \\ +3t^2 \end{array}}$$

$$1 + 3t + 7t^2 + 3t^3 + t^4 = (1+t)^4 - t(1+t)^2 + 3t^2$$

The γ-vector

If $h(t) = \sum_{i=0}^{n} h_i t^i$ is symmetric, then there exist γ_i such that

$$h(t) = \sum_{i=0}^{n/2} \gamma_i t^i (1+t)^{n-2i},$$

e.g.,

$$\begin{array}{c} 1 + 3t + 7t^2 + 3t^3 + t^4 \\ \hline 1 + 4t + 6t^2 + 4t^3 + t^4 \\ -t(1 + 2t + t^2) \\ +3t^2 \end{array}$$

$$1 + 3t + 7t^2 + 3t^3 + t^4 = (1+t)^4 - t(1+t)^2 + 3t^2$$

the vector $(\gamma_0, \gamma_1, \ldots)$ is called the γ-*vector*

Gal's conjecture

For Δ a sphere, Dehn-Sommerville relations say $h(\Delta)$ is symmetric,

$$h_i = h_{n-i},$$

and hence Δ has a well-defined γ-vector, denoted $\gamma(\Delta)$

Gal's conjecture

For Δ a sphere, Dehn-Sommerville relations say $h(\Delta)$ is symmetric,

$$h_i = h_{n-i},$$

and hence Δ has a well-defined γ-vector, denoted $\gamma(\Delta)$

Conjecture (Gal (2005))

If Δ is a flag homology sphere, then $\gamma(\Delta)$ is nonnegative

Gal's conjecture

For Δ a sphere, Dehn-Sommerville relations say $h(\Delta)$ is symmetric,

$$h_i = h_{n-i},$$

and hence Δ has a well-defined γ-vector, denoted $\gamma(\Delta)$

Conjecture (Gal (2005))

If Δ is a flag homology sphere, then $\gamma(\Delta)$ is nonnegative

- implies the Charney-Davis conjecture

Gal's conjecture

For Δ a sphere, Dehn-Sommerville relations say $h(\Delta)$ is symmetric,

$$h_i = h_{n-i},$$

and hence Δ has a well-defined γ-vector, denoted $\gamma(\Delta)$

Conjecture (Gal (2005))

If Δ is a flag homology sphere, then $\gamma(\Delta)$ is nonnegative

- implies the Charney-Davis conjecture
- true in dimension ≤ 4 and other interesting cases (e.g., barycentric subdivisions, Coxeter complexes)

Gal's conjecture

For Δ a sphere, Dehn-Sommerville relations say $h(\Delta)$ is symmetric,

$$h_i = h_{n-i},$$

and hence Δ has a well-defined γ-vector, denoted $\gamma(\Delta)$

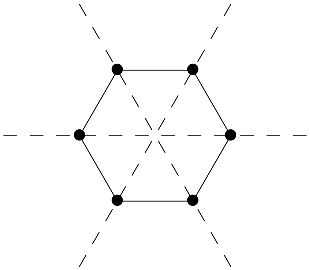
Conjecture (Gal (2005))

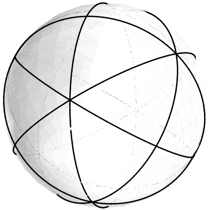
If Δ is a flag homology sphere, then $\gamma(\Delta)$ is nonnegative

- implies the Charney-Davis conjecture
- true in dimension ≤ 4 and other interesting cases (e.g., barycentric subdivisions, Coxeter complexes)

What do the entries of the γ-vector count?

Combinatorial interpretations for γ-vectors





Eulerian polynomials

Let $w \in S_{n+1}$ and $d(w) := |\{i : w_i > w_{i+1}\}|$. Then,

$$A_n(t) = \sum_{w \in S_{n+1}} t^{d(w)} = h(\Delta(A_n); t)$$

Eulerian polynomials

Let $w \in S_{n+1}$ and $d(w) := |\{i : w_i > w_{i+1}\}|$. Then,

$$A_n(t) = \sum_{w \in S_{n+1}} t^{d(w)} = h(\Delta(A_n); t)$$

w	$d(w)$	$t^{d(w)}$
123		
132		
213		
231		
312		
321		

Eulerian polynomials

Let $w \in S_{n+1}$ and $d(w) := |\{i : w_i > w_{i+1}\}|$. Then,

$$A_n(t) = \sum_{w \in S_{n+1}} t^{d(w)} = h(\Delta(A_n); t)$$

w	$d(w)$	$t^{d(w)}$
123	0	1
132		
213		
231		
312		
321		

Eulerian polynomials

Let $w \in S_{n+1}$ and $d(w) := |\{i : w_i > w_{i+1}\}|$. Then,

$$A_n(t) = \sum_{w \in S_{n+1}} t^{d(w)} = h(\Delta(A_n); t)$$

w	$d(w)$	$t^{d(w)}$
123	0	1
132	1	t
213		
231		
312		
321		

Eulerian polynomials

Let $w \in S_{n+1}$ and $d(w) := |\{i : w_i > w_{i+1}\}|$. Then,

$$A_n(t) = \sum_{w \in S_{n+1}} t^{d(w)} = h(\Delta(A_n); t)$$

w	$d(w)$	$t^{d(w)}$
123	0	1
132	1	t
213	1	t
231		
312		
321		

Eulerian polynomials

Let $w \in S_{n+1}$ and $d(w) := |\{i : w_i > w_{i+1}\}|$. Then,

$$A_n(t) = \sum_{w \in S_{n+1}} t^{d(w)} = h(\Delta(A_n); t)$$

w	$d(w)$	$t^{d(w)}$
123	0	1
132	1	t
213	1	t
231	1	t
312		
321		

Eulerian polynomials

Let $w \in S_{n+1}$ and $d(w) := |\{i : w_i > w_{i+1}\}|$. Then,

$$A_n(t) = \sum_{w \in S_{n+1}} t^{d(w)} = h(\Delta(A_n); t)$$

w	$d(w)$	$t^{d(w)}$
123	0	1
132	1	t
213	1	t
231	1	t
312	1	t
321		

Eulerian polynomials

Let $w \in S_{n+1}$ and $d(w) := |\{i : w_i > w_{i+1}\}|$. Then,

$$A_n(t) = \sum_{w \in S_{n+1}} t^{d(w)} = h(\Delta(A_n); t)$$

w	$d(w)$	$t^{d(w)}$
123	0	1
132	1	t
213	1	t
231	1	t
312	1	t
321	2	t^2

Eulerian polynomials

Let $w \in S_{n+1}$ and $d(w) := |\{i : w_i > w_{i+1}\}|$. Then,

$$A_n(t) = \sum_{w \in S_{n+1}} t^{d(w)} = h(\Delta(A_n); t)$$

w	$d(w)$	$t^{d(w)}$
123	0	1
132	1	t
213	1	t
231	1	t
312	1	t
321	2	t^2

$$A_2(t) = 1 + 4t + t^2 = h(\Delta(A_2); t)$$

Eulerian polynomials

We have:

$$A_1(t) = 1 + t$$

$$A_2(t) = 1 + 4t + t^2$$

$$A_3(t) = 1 + 11t + 11t^2 + t^3$$

$$A_4(t) = 1 + 26t + 66t^2 + 26t^3 + t^4$$

$$\vdots$$

Eulerian polynomials

We have:

$$\begin{aligned} A_1(t) &= 1 + t \\ &= (1+t) \\ A_2(t) &= 1 + 4t + t^2 \\ \\ A_3(t) &= 1 + 11t + 11t^2 + t^3 \\ \\ A_4(t) &= 1 + 26t + 66t^2 + 26t^3 + t^4 \\ &\vdots \end{aligned}$$

Eulerian polynomials

We have:

$$\begin{aligned} A_1(t) &= 1 + t \\ &= (1 + t) \\ A_2(t) &= 1 + 4t + t^2 \\ &= (1 + t)^2 + 2t \\ A_3(t) &= 1 + 11t + 11t^2 + t^3 \\ \\ A_4(t) &= 1 + 26t + 66t^2 + 26t^3 + t^4 \\ \\ &\vdots \end{aligned}$$

Eulerian polynomials

We have:

$$\begin{aligned} A_1(t) &= 1 + t \\ &= (1 + t) \\ A_2(t) &= 1 + 4t + t^2 \\ &= (1 + t)^2 + 2t \\ A_3(t) &= 1 + 11t + 11t^2 + t^3 \\ &= (1 + t)^3 + 8t(1 + t) \\ A_4(t) &= 1 + 26t + 66t^2 + 26t^3 + t^4 \end{aligned}$$

$$\vdots$$

Eulerian polynomials

We have:

$$\begin{aligned}
A_1(t) &= 1 + t \\
&= (1 + t) \\
A_2(t) &= 1 + 4t + t^2 \\
&= (1 + t)^2 + 2t \\
A_3(t) &= 1 + 11t + 11t^2 + t^3 \\
&= (1 + t)^3 + 8t(1 + t) \\
A_4(t) &= 1 + 26t + 66t^2 + 26t^3 + t^4 \\
&= (1 + t)^4 + 22t(1 + t)^2 + 16t^2 \\
&\vdots
\end{aligned}$$

The γ-vector for A_n

Define

$$\widehat{S}_n = \{w \in S_n : w_{n-1} < w_n, \text{ and if } w_{i-1} > w_i \text{ then } w_i < w_{i-1}\}$$

The γ-vector for A_n

Define

$$\widehat{S}_n = \{w \in S_n : w_{n-1} < w_n, \text{ and if } w_{i-1} > w_i \text{ then } w_i < w_{i-1}\}$$

Theorem (Foata-Schützenberger (1970))

$$A_n(t) = \sum_{w \in \widehat{S}_{n+1}} t^{d(w)}(1+t)^{n-2d(w)},$$

The γ-vector for A_n

Define

$$\widehat{S}_n = \{w \in S_n : w_{n-1} < w_n, \text{ and if } w_{i-1} > w_i \text{ then } w_i < w_{i-1}\}$$

Theorem (Foata-Schützenberger (1970))

$$A_n(t) = \sum_{w \in \widehat{S}_{n+1}} t^{d(w)}(1+t)^{n-2d(w)},$$

i.e.,

$$\gamma_i(A_n) = |\{w \in \widehat{S}_{n+1} : d(w) = i\}|$$

The γ-vector for A_n

$$A_n(t) = \sum_{w \in \widehat{S}_{n+1}} t^{d(w)}(1+t)^{n-2d(w)}$$

The γ-vector for A_n

$$A_n(t) = \sum_{w \in \widehat{S}_{n+1}} t^{d(w)}(1+t)^{n-2d(w)}$$

w	$d(w)$	$t^{d(w)}(1+t)^{n-2d(w)}$
1234		
1324		
1423		
2134		
2314		
2413		
3124		
3412		
4123		

The γ-vector for A_n

$$A_n(t) = \sum_{w \in \widehat{S}_{n+1}} t^{d(w)}(1+t)^{n-2d(w)}$$

w	$d(w)$	$t^{d(w)}(1+t)^{n-2d(w)}$
1234	0	$(1+t)^3$
1324		
1423		
2134		
2314		
2413		
3124		
3412		
4123		

The γ-vector for A_n

$$A_n(t) = \sum_{w \in \widehat{S}_{n+1}} t^{d(w)}(1+t)^{n-2d(w)}$$

w	$d(w)$	$t^{d(w)}(1+t)^{n-2d(w)}$
1234	0	$(1+t)^3$
1324	1	$t(1+t)$
1423		
2134		
2314		
2413		
3124		
3412		
4123		

The γ-vector for A_n

$$A_n(t) = \sum_{w \in \widehat{S}_{n+1}} t^{d(w)}(1+t)^{n-2d(w)}$$

w	$d(w)$	$t^{d(w)}(1+t)^{n-2d(w)}$
1234	0	$(1+t)^3$
1324	1	$t(1+t)$
1423	1	$t(1+t)$
2134		
2314		
2413		
3124		
3412		
4123		

The γ-vector for A_n

$$A_n(t) = \sum_{w \in \widehat{S}_{n+1}} t^{d(w)}(1+t)^{n-2d(w)}$$

w	$d(w)$	$t^{d(w)}(1+t)^{n-2d(w)}$
1234	0	$(1+t)^3$
1324	1	$t(1+t)$
1423	1	$t(1+t)$
2134	1	$t(1+t)$
2314		
2413		
3124		
3412		
4123		

The γ-vector for A_n

$$A_n(t) = \sum_{w \in \widehat{S}_{n+1}} t^{d(w)}(1+t)^{n-2d(w)}$$

w	$d(w)$	$t^{d(w)}(1+t)^{n-2d(w)}$
1234	0	$(1+t)^3$
1324	1	$t(1+t)$
1423	1	$t(1+t)$
2134	1	$t(1+t)$
2314	1	$t(1+t)$
2413		
3124		
3412		
4123		

The γ-vector for A_n

$$A_n(t) = \sum_{w \in \widehat{S}_{n+1}} t^{d(w)}(1+t)^{n-2d(w)}$$

w	$d(w)$	$t^{d(w)}(1+t)^{n-2d(w)}$
1234	0	$(1+t)^3$
1324	1	$t(1+t)$
1423	1	$t(1+t)$
2134	1	$t(1+t)$
2314	1	$t(1+t)$
2413	1	$t(1+t)$
3124		
3412		
4123		

The γ-vector for A_n

$$A_n(t) = \sum_{w \in \widehat{S}_{n+1}} t^{d(w)}(1+t)^{n-2d(w)}$$

w	$d(w)$	$t^{d(w)}(1+t)^{n-2d(w)}$
1234	0	$(1+t)^3$
1324	1	$t(1+t)$
1423	1	$t(1+t)$
2134	1	$t(1+t)$
2314	1	$t(1+t)$
2413	1	$t(1+t)$
3124	1	$t(1+t)$
3412		
4123		

The γ-vector for A_n

$$A_n(t) = \sum_{w \in \widehat{S}_{n+1}} t^{d(w)}(1+t)^{n-2d(w)}$$

w	$d(w)$	$t^{d(w)}(1+t)^{n-2d(w)}$
1234	0	$(1+t)^3$
1324	1	$t(1+t)$
1423	1	$t(1+t)$
2134	1	$t(1+t)$
2314	1	$t(1+t)$
2413	1	$t(1+t)$
3124	1	$t(1+t)$
3412	1	$t(1+t)$
4123		

The γ-vector for A_n

$$A_n(t) = \sum_{w \in \widehat{S}_{n+1}} t^{d(w)}(1+t)^{n-2d(w)}$$

w	$d(w)$	$t^{d(w)}(1+t)^{n-2d(w)}$
1234	0	$(1+t)^3$
1324	1	$t(1+t)$
1423	1	$t(1+t)$
2134	1	$t(1+t)$
2314	1	$t(1+t)$
2413	1	$t(1+t)$
3124	1	$t(1+t)$
3412	1	$t(1+t)$
4123	1	$t(1+t)$

The γ-vector for A_n

$$A_n(t) = \sum_{w \in \widehat{S}_{n+1}} t^{d(w)}(1+t)^{n-2d(w)}$$

w	$d(w)$	$t^{d(w)}(1+t)^{n-2d(w)}$
1234	0	$(1+t)^3$
1324	1	$t(1+t)$
1423	1	$t(1+t)$
2134	1	$t(1+t)$
2314	1	$t(1+t)$
2413	1	$t(1+t)$
3124	1	$t(1+t)$
3412	1	$t(1+t)$
4123	1	$t(1+t)$

$$A_3(t) = (1+t)^3 + 8t(1+t) = 1 + 11t + 11t^2 + t^3$$

Kruskal-Katona inequalities

Observe that

$$\gamma(A_1) = (1)$$
$$\gamma(A_2) = (1, 2)$$
$$\gamma(A_3) = (1, 8)$$
$$\gamma(A_4) = (1, 22, 16)$$
$$\vdots$$

are all *Kruskal-Katona vectors*

Combinatorial interpretations for γ-vectors

Gal's conjecture

An example

The Γ complex

A conjecture

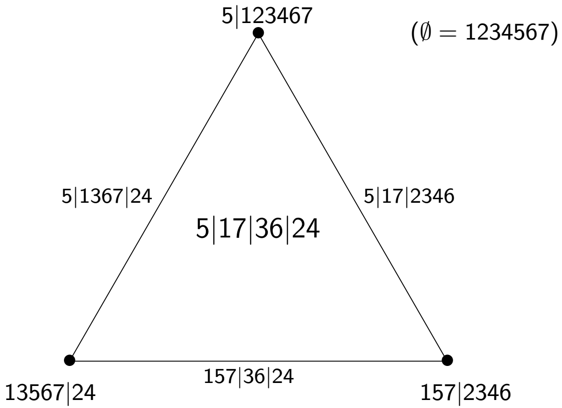
The complex $\Gamma(A_n)$

We can identify elements of $\widehat{S}_{n+1}$ with faces a simplicial complex, denoted $\Gamma(A_n)$

The complex $\Gamma(A_n)$

We can identify elements of $\widehat{S}_{n+1}$ with faces a simplicial complex, denoted $\Gamma(A_n)$

A facet of $\Gamma(A_6)$:



The Γ complex

Theorem (Nevo-P.)

There exists a simplicial complex $\Gamma(\Delta)$ *such that*

$$\gamma(\Delta) = f(\Gamma(\Delta))$$

for the following flag spheres:

The Γ complex

Theorem (Nevo-P.)

There exists a simplicial complex $\Gamma(\Delta)$ *such that*

$$\gamma(\Delta) = f(\Gamma(\Delta))$$

for the following flag spheres:

- Δ *is a Coxeter complex*

The Γ complex

Theorem (Nevo-P.)

There exists a simplicial complex $\Gamma(\Delta)$ *such that*

$$\gamma(\Delta) = f(\Gamma(\Delta))$$

for the following flag spheres:

- Δ *is a Coxeter complex*
- Δ *is (dual to) an associahedron*

The Γ complex

Theorem (Nevo-P.)

There exists a simplicial complex $\Gamma(\Delta)$ *such that*

$$\gamma(\Delta) = f(\Gamma(\Delta))$$

for the following flag spheres:

- Δ *is a Coxeter complex*
- Δ *is (dual to) an associahedron*
- Δ *is (dual to) a cyclohedron*

The Γ complex

Theorem (Nevo-P.)

There exists a simplicial complex $\Gamma(\Delta)$ *such that*

$$\gamma(\Delta) = f(\Gamma(\Delta))$$

for the following flag spheres:

- Δ *is a Coxeter complex*
- Δ *is (dual to) an associahedron*
- Δ *is (dual to) a cyclohedron*
- Δ *has at most* $2n+3$ *vertices*

The Γ complex

Theorem (Nevo-P.)

There exists a simplicial complex $\Gamma(\Delta)$ *such that*

$$\gamma(\Delta) = f(\Gamma(\Delta))$$

for the following flag spheres:

- Δ *is a Coxeter complex*
- Δ *is (dual to) an associahedron*
- Δ *is (dual to) a cyclohedron*
- Δ *has at most* $2n+3$ *vertices*
- *[P.-Tenner]* Δ *is a barycentric subdivision of with* $\dim \Delta \leq 8$ *(builds on Brenti-Welker)*

Combinatorial interpretations for γ-vectors

A conjecture - what γ counts?

Conjecture (Nevo-P.)

If Δ is a flag homology sphere, then $\gamma(\Delta) = f(\Gamma(\Delta))$ for some (flag? balanced?) simplicial complex $\Gamma(\Delta)$

Questions?

"On γ-vectors satisfying the Kruskal-Katona inequalities," with E. Nevo, *Discrete and Computational Geometry*, to appear.
arXiv:0909.0694