Denominator identities and Lie superalgebras

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We find an analogue of the Weyl denominator identity for a basic classical Lie superalgebra.

joint work with Victor Kac and Pierluigi Möseneder Frajria
Weyl denominator identity

\[ g \text{ complex finite-dimensional Lie algebra} \]
\[ h \text{ Cartan subalgebra} \]
\[ \Delta \text{ root system of } (g, h) \]
\[ W \text{ Weyl group of } \Delta \]
\[ \Delta^+ \subset \Delta \text{ set of positive roots, } \rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha \]

Theorem

\[ \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha}) = \sum_{w \in W} sgn(w) e^{w(\rho) - \rho} \]

For \( g = sl(n) \), Weyl formula is related to the expansion of the Vandermonde determinant.
Basic classical Lie superalgebras

First we want to locate a suitable class of Lie superalgebras to which trying to extend the Weyl formula. These are the

Basic classical Lie superalgebras $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$:

- $\mathfrak{g}$ is simple;
- $\mathfrak{g}_0$ is a reductive Lie algebra;
- $\mathfrak{g}$ has a nondegenerate bilinear invariant form which is symmetric on $\mathfrak{g}_0$, symplectic on $\mathfrak{g}_1$ and such that $(\mathfrak{g}_0, \mathfrak{g}_1) = 0$.

$\mathfrak{h} \subset \mathfrak{g}_0$ Cartan subalgebra

$\mathfrak{g} = \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ weight space decomposition

$\Delta = \Delta_0 \cup \Delta_1$ decomposition into even and odd “roots”

$\mathcal{W}$ Weyl group of $\Delta_0$. 

Example: $gl(m, n)$.

$$gl(m, n) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right\}, \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

is a Lie superalgebra w.r.t. $[X, Y] = XY - (-1)^{\text{deg}(X)\text{deg}(Y)} YX$.

$h = \text{diagonal matrices}, h^* = \bigoplus_{i=1}^{m+n} \mathbb{C} \varepsilon_i, \delta_j = \varepsilon_{m+j}, 1 \leq j \leq n.$

$$(X, Y) = \text{str}(XY), \quad \text{str}\left( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) = \text{tr}(A) - \text{tr}(D).$$

The above bilinear form can be normalized in such a way that

$$(\varepsilon_i, \varepsilon_j) = \delta_{ij}, \quad (\delta_h, \delta_k) = -\delta_{hk}.$$

$$\Delta_0 = \pm\{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq m\} \cup \pm\{\delta_i - \delta_j \mid 1 \leq i < j \leq n\}$$

$$\Delta_1 = \pm\{\varepsilon_i - \delta_j \mid 1 \leq i \leq m, 1 \leq j \leq n\}$$
Why Weyl formula doesn’t work for superalgebras....

Because:
1. the restriction of $(\cdot, \cdot)$ to $\text{Span}_\mathbb{R} \Delta$ is usually indefinite;
2. the sets of positive roots are no more $\mathcal{W}$-conjugate.

Corresponding comments:
1. One defines the defect $d$ of $\mathfrak{g}$ as the dimension of a maximal isotropic subspace of $\sum_{\alpha \in \Delta} \mathbb{R} \alpha$. It can be shown that $d$ equals the cardinality of a maximal isotropic subset of $\Delta^+$ (a subset $S \subset \Delta^+$ is isotropic if it is formed by linearly independent pairwise orthogonal isotropic roots);
2. it won’t be a surprise that formulas do depend on the choice of $\Delta^+$. 
Kac-Wakimoto-Gorelik Theorem

Fix $\Delta^+$. Define the Weyl-Kac denominator and superdenominator

$$R^+ = \frac{\prod_{\alpha \in \Delta_0^+} (1 - e^{-\alpha})}{\prod_{\alpha \in \Delta_1^+} (1 + e^{-\alpha})}, \quad R^- = \frac{\prod_{\alpha \in \Delta_0^+} (1 - e^{-\alpha})}{\prod_{\alpha \in \Delta_1^+} (1 - e^{-\alpha})}$$

where $\rho = \rho_0 - \rho_1$, $\rho_i = \frac{1}{2} \sum_{\alpha \in \Delta_i^+} \alpha$, $i = 0, 1$.

Theorem

Let $\Delta^+$ be any set of positive roots such that a maximal isotropic subset $S$ of $\Delta^+$ is contained in the set of simple roots $\Pi$ corresponding to $\Delta^+$. Then

$$e^\rho R^\pm = \sum_{w \in W^\#} sgn^\pm(w) w \frac{e^\rho}{\prod_{\beta \in S} (1 \pm e^{-\beta})}.$$ 

where $W^\#$ is a subgroup of $W$. 
**Main result**

**Definition**

A set of positive root $\Delta^+$ is called distinguished if the corresponding set of simple roots has exactly one odd root.

The following result is proven in the extended abstract, by representation theoretic methods.

**Theorem**

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a basic classical Lie superalgebra of defect $d$, where $\mathfrak{g} = A(d - 1, d - 1)$ is replaced by $\text{gl}(d, d)$. Then, for any distinguished set of positive roots, we have

$$C e^\rho R^\pm = \sum_{w \in W} \text{sgn}(w) w \frac{e^\rho}{(1 \pm e^{-\gamma_1})(1 - e^{-\gamma_1-\gamma_2}) \cdots (1 + (-1)^{d+1} e^{-\gamma_1-\gamma_2-\cdots-\gamma_d})}$$

where $\{\gamma_1, \ldots, \gamma_d\}$ is an explicitly defined maximal isotropic subset of $\Delta^+$ and $C$ is an explicit constant.
• Distinguished sets of positive root exist for any $\mathfrak{g}$ and have been classified by Kac.

• For a distinguished $\Delta^+$ the hypothesis of KWG Theorem holds iff $d = 1$. This is a strong constraint, since, e.g.

$$\text{def } gl(m, n) = \min\{m, n\}.$$ 

• Distinguished sets of positive root are quite often used when presenting $\mathfrak{g}$ as a contragredient Lie superalgebra (roughly speaking: by generators and relations).

• As we shall see, the proof follows quite naturally by Howe duality (and for type $A$, in a more elementary way, by Cauchy formulas).
The metaplectic representation

The choice of a set of positive roots $\Delta^+$ determines a polarization $g_1 = g_1^+ + g_1^-$, where $g_1^\pm = \bigoplus_{\alpha \in \Delta_1^\pm} g_\alpha$.

Hence we can consider the Weyl algebra

$$W(g_1) = T(g_1) / \langle x \otimes y - y \otimes x - (x, y) \rangle$$

of $(g_1, (\ , \ )_{g_1})$ and construct the left $W(g_1)$-module

$$M^{\Delta^+}(g_1) = W(g_1) / W(g_1)g_1^+,$$

The module $M^{\Delta^+}(g_1)$ is also a $sp(g_1, (\ , \ ))$–module with $T \in sp(g_1, (\ , \ ))$ acting by left multiplication by

$$\theta(T) = -\frac{1}{2} \sum_{i=1}^{\dim g_1} T(x_i)x^i,$$

where $\{x_i\}$ is any basis of $g_1$ and $\{x^i\}$ is its dual basis w.r.t. $(\ , \ )$. Since $ad(g_0) \subset sp(g_1, (\ , \ ))$, we obtain an action of $g_0$ on $M^{\Delta^+}(g_1)$. 
The metaplectic representation

We have a $\mathfrak{h}$-module isomorphism $M^{\Delta^+}(g_1) \cong S(g_1^-) \otimes \mathbb{C}_{-\rho_1}$ where $S(g_1^-)$ is the symmetric algebra of $g_1^-$. Its $\mathfrak{h}$-character is

$$chM^{\Delta^+}(g_1) = \frac{e^{-\rho_1}}{\prod_{\alpha \in \Delta_1^+} (1 - e^{-\alpha})}. \quad (1)$$

Upon multiplication by $e^{\rho_0} \prod_{\alpha \in \Delta_0^+} (1 - e^{-\alpha})$ the r.h.s. of (1) becomes $e^{\rho} R^-$

So if we are able to determine the $g_0$-character of $M^{\Delta^+}(g_1)$ we have a formula for $R^-$. 
Example: $sl(m, n)$ and Cauchy formulas

In this case there is essentially a unique distinguished set of positive roots: $\mathfrak{g}_1^+ = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$. The special feature of this example is that $\mathfrak{g}_1^-$ is a $\mathfrak{g}_0$-module, and the action of $\mathfrak{g}_0$ on $\mathfrak{g}_1^-$ is the natural action of $\{(A, B) \in gl(m + 1) \times gl(n + 1) \mid tr(A) + tr(B) = 0\}$ on $(\mathbb{C}^{m+1})^* \otimes \mathbb{C}^{n+1}$. Assume $m > n$. Cauchy formulas give

$$ch(S(\mathfrak{g}_1^-)) = ch(S((\mathbb{C}^{m+1})^* \otimes \mathbb{C}^{n+1})) = \sum_{\lambda} L^{A_m}(\tau(\lambda))L^{A_n}(\lambda)$$

where for $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{n+1}$ we have set

$$\lambda = \sum_{i=1}^{n+1} \lambda_i \delta_i, \quad \tau(\lambda) = -w_0(\sum_{i=1}^{n+1} \lambda_i \varepsilon_i)$$

and $w_0$ is the longest element in the symmetric group $W(A_m)$. 
Then

\[ S(g_1^0) \otimes C_{-\rho_1} = \bigoplus_{s_1 \geq s_2 \geq \ldots \geq s_{n+1}} L^{A_m \times A_n}(-\rho_1 - s_1 \gamma_1 - \ldots - s_{n+1} \gamma_{n+1}), \]

\[ \gamma_1 = \varepsilon_{m+1} - \delta_1, \quad \gamma_2 = \varepsilon_{m} - \delta_2, \ldots \ldots, \gamma_{n+1} = \varepsilon_{m-n+1} - \delta_{n+1}. \]

By the Weyl character formula, we have for \( s = s_1 \geq s_2 \geq \ldots \geq s_{n+1} \)

\[ \frac{e^{-\rho_1}}{\prod_{\beta \in \Delta_1^+} (1 - e^{-\beta})} = \sum_s \text{ch} L^{A_m \times A_n}(\lambda_{s_1}, \ldots, s_{n+1}) = \sum_s \sum_{w \in W} \text{sgn}(w) \frac{e^{w(\lambda_{s_1}, \ldots, s_{n+1} + \rho_0)}}{e^{\rho_0} \prod_{\beta \in \Delta_0^+} (1 - e^{-\alpha})}. \]

Hence

\[ e^\rho R^- = \sum_{s_1 \geq s_2 \geq \ldots \geq s_{n+1}} \sum_{w \in W} \text{sgn}(w) e^{w(\rho - s_1 \gamma_1 - \ldots - s_{n+1} \gamma_{n+1})} \]

\[ \sum_{s_1, s_2, \ldots, s_{n+1}} \sum_{w \in W} \text{sgn}(w) e^{w(\rho - s_1 \gamma_1 - s_2 (\gamma_1 + \gamma_2) - \ldots - s_{n+1} (\gamma_1 + \ldots + \gamma_{n+1}))} \]
General case: Howe duality

For a distinguished set of positive roots, we build up a real form $V$ of $g_1$ endowed with a standard symplectic basis $\{e_\alpha, f_\alpha\}_{\alpha \in \Delta_1^+}$ such that $\bigoplus_{\alpha \in \Delta_1^+} \mathbb{C}(e_\alpha \pm \sqrt{-1}f_\alpha) = g_1^+$. It turns out that $sp(V) \cap ad(g_0) = s_1 \times s_2$, $s_i$, $i = 1, 2$, being the Lie algebras of a compact dual pair $(G_1, G_2)$ in $Sp(V)$. Distinguished sets of positive roots turn out to correspond in this way to compact dual pairs $(G_1, G_2)$, with $G_1$ compact:

$\Delta_B^+ \rightarrow (O(2m+1), Sp(2n, \mathbb{R})), \quad \Delta_A^+ \rightarrow (U(m), U(n)),$

$\Delta_{D1}^+ \rightarrow (O(2m), Sp(2n, \mathbb{R})), \quad \Delta_{D2}^+ \rightarrow (Sp(m), SO^*(2n)).$

If $\lambda \rightarrow \tau(\lambda)$ is the Theta correspondence, then, as $g_0$-modules

$$ch M^{\Delta^+}(g_1) = \sum_\lambda L^{G_1}(\lambda)L^{(s_2)\mathbb{C}}(\tau(\lambda)).$$
By now we have proven WITH COMBINATORIAL METHODS the following theorem.

**Theorem**

Let $g = g_0 \oplus g_1$ be a basic classical Lie superalgebra of defect $d$, where $g = A(d-1, d-1)$ is replaced by $gl(d, d)$. Let $\Delta^+$ be any set of positive roots and $Q^+$ be the corresponding positive root lattice. There is a combinatorial procedure that starting from $\Delta^+$ yields a class of maximal isotropic sets in $\Delta^+$. Fix any of them, say $S_{\text{special}} = \{\gamma_1, \ldots, \gamma_d\}$. Then we have

$$C \cdot e^{\rho} R^\pm = \sum_{w \in W_g} \text{sgn}^\pm(w) w \frac{e^{\rho}}{\prod_{i=1}^{d} (1 \pm \chi_i^{\pm} e^{-\langle \gamma_i \rangle})}$$

for an explicit constant $C$. Moreover, there exists a choice for $S_{\text{special}}$ such that $\langle \gamma_i \rangle \in Q^+$ for any $i = 1, \ldots, d$. 

**Updating the main result...**
Undefined notation:

\[ \varepsilon(\eta) = \begin{cases} 1 & \text{if } \eta \in \mathbb{Z}\Delta_0, \\ -1 & \text{if } \eta \in \mathbb{Z}\Delta \setminus \mathbb{Z}\Delta_0, \end{cases} \]

\[ \gamma_i^{\leq} = \{ \beta \in S^{\text{special}}, \beta \leq \gamma_i \}, \]

\[ \langle \gamma_i \rangle = \sum_{\beta \in \gamma_i^{\leq}} \varepsilon(\gamma_i - \beta)\beta, \]

\[ \chi_i^{+} = (-1)^{|\gamma_i^{\leq}|+1}, \chi_i^{-} = 1 \]

where \( \beta \leq \gamma \) if \( \gamma - \beta \) is a sum of positive roots or zero.

A nice feature of the theorem is that it allows us to recover the Theta correspondence for compact dual pairs starting from the combinatorial formula. This is highly non trivial, but it works!