Bruhat order, rationally smooth Schubert varieties, and hyperplane arrangements

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Abstract

We link Schubert varieties in the generalized flag manifolds with hyperplane arrangements. For an element of a Weyl group, we construct a certain graphical hyperplane arrangement. We show that the generating function for regions of this arrangement coincides with the Poincaré polynomial of the corresponding Schubert variety if and only if the Schubert variety is rationally smooth.

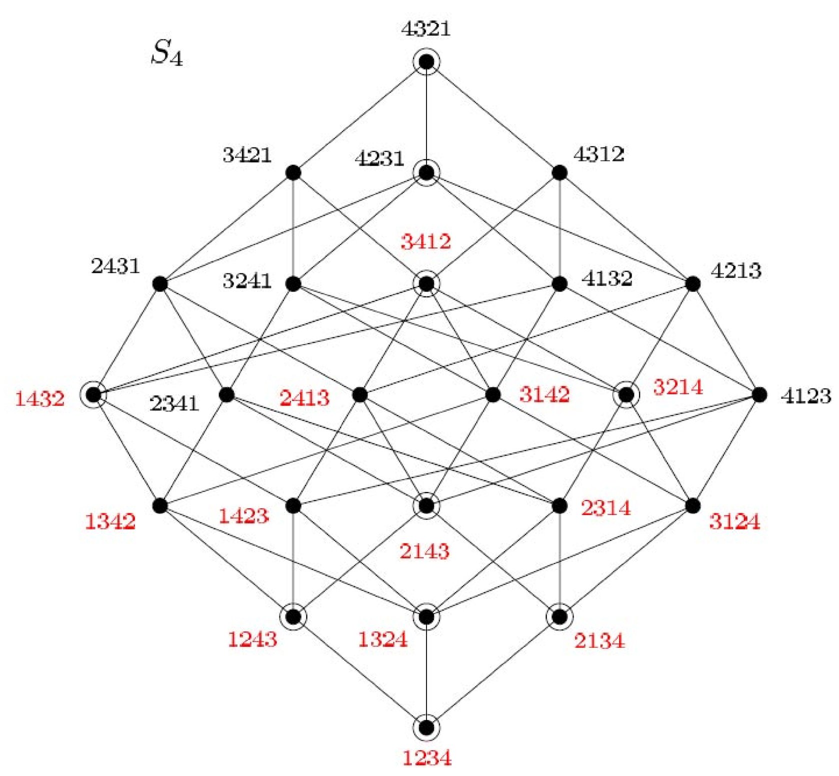
1. The basic idea and the Motivation

- Given an element w of W(a Weyl group), we are going to study two polynomials P_w and R_w.
- P_w(the Poincaré polynomial) comes from a Schubert variety X_w.
- R_w comes from a hyperplane arrangement A_w.
- So they are very different in nature.
- They are equal if and only if a geometric condition of X_w is satisfied : when X_w is rationally smooth.
- What we do is purely combinatorial, since
  - P_w can be defined purely combinatorially and,
  - the geometric condition above can be described as pattern avoidance conditions for $w \in W$.

2. P_w : the Poincaré Polynomial coming from X_w : the Schubert variety

- (Not needed) Schubert variety $X_w := \overline{BwB}/B$.
- Lower Bruhat interval $[\mathrm{id}, w] := \{u \in W \mid u \leq w\}$.
- P_w : the rank generating function for $[\mathrm{id}, w]$. Here the rank is the number of inversions.
- (Carrell-Peterson Criteria) X_w is rationally smooth iff P_w is palindromic(Symmetric coefficients).
- By definition, we have $P_{w^{-1}} = P_w$.

Example : P_w in A_4

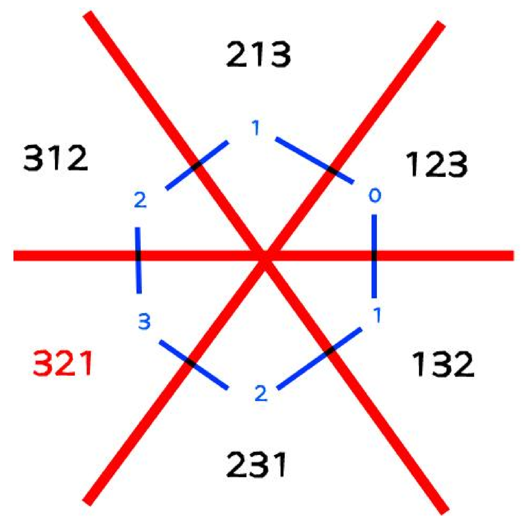


- $P_{id} = 1$.
- $P_{2143} = q^2 + 2q + 1$
- $P_{3412} = q^3 + 4q^2 + 5q^2 + 3q + 1$: Not palindromic!
- $P_{4321} = q^6 + 3q^5 + 5q^4 + 6q^3 + 5q^2 + 3q + 1$

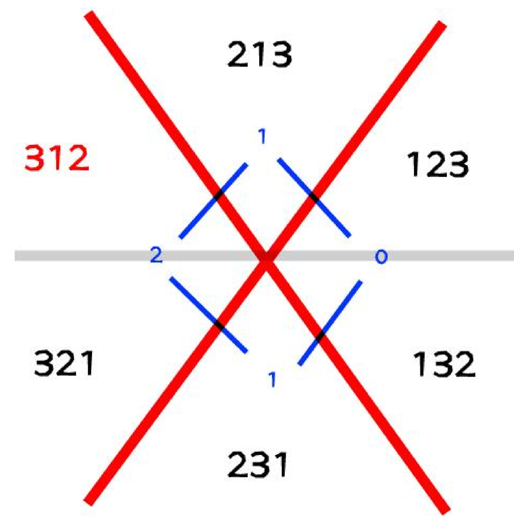
3. R_w coming from A_w : the inversion hyperplane arrangement

- $\mathcal{A}_w$ is the collection of hyperplanes $\alpha(x) = 0$ where $\alpha > 0$ s.t. $w(\alpha) < 0$.
- This is the generalization of the Coxeter arrangement, where we only take hyperplanes corresponding to inversions of w.
- We define the distance enumerator polynomial R_w:
  - For regions r, r' of $\mathcal{A}_w$, $d(r, r') :=$ Minimal # of hyperplanes crossed to go from r to r'.
  - r_0 : Region containing the fundamental chamber.
  - R_w : the rank generating function for regions of A_w. Here the rank is the distance from r_0.
- R_w is always palindromic.
- $R_{w^{-1}} = R_w$.

Example : R_w in type A_2



$R_{321} = q^3 + 2q^2 + 2q + 1$



$R_{312} = q^2 + 2q + 1$

Combinatorial restatement of our main theorem

$R_w = P_w$ if and only if P_w is palindromic.

4. Sketch of the Main idea of the proof

- (Billey-Postnikov) When $w \in W$ is rationally smooth, either w or w^{-1} decomposes as uv where $u \in W_J, v \in W^J$ such that
  - u is maximal element of W_J below w (or w^{-1}),
  - J corresponds to leaf-removed subset of Dynkin diagram of W.
- Then we get a factorization : $P_w = P_u P_v^{W^J}$

Key idea

v has palindromic lower interval in W^J if and only if the interval is isomorphic to some maximal parabolic quotient of some Weyl group.

- Using this, we can decompose further : $w = u_1 u_2 v$ so that u_2 is the longest element of $W_{I \cap J}$ and I is the set of simple reflections in v.
- Then A_{u_1} divides A_w nicely, and we get $R_w / R_u = R_{u_2 v} / R_{u_2}$.
- Next we show that $R_{u_2 v} / R_{u_2} = P_v^{W^J}$.
- Hence P_w and R_w factorizes similarly if w is rationally smooth.

5. Conclusion and further remarks

Main theorem

$P_w = R_w$ if and only if X_w is rationally smooth.

- It would be interesting to check if our statement is also true for Coxeter groups in general.
- Whenever $P_w = R_w$, the q-factors of P_w are exactly the roots of the characteristic polynomial of A_w. Is it true for non-rationally smooth w?
- Is there a bijective proof for our statement?