Abstract

We link Schubert varieties in the generalized flag manifolds with hyperplane arrangements. For an element of a Weyl group, we construct a certain graphical hyperplane arrangement. We show that the generating function for regions of this arrangement coincides with the Poincaré polynomial of the corresponding Schubert variety if and only if the Schubert variety is rationally smooth.

1. The basic idea and the Motivation

- Given an element \( w \) of \( W \) (a Weyl group), we are going to study two polynomials \( P_w \) and \( R_w \).
- \( P_w \) (the Poincaré polynomial) comes from a Schubert variety \( X_w \).
- \( R_w \) comes from a hyperplane arrangement \( A_w \).
- So they are very different in nature.
- They are equal if and only if a geometric condition of \( X_w \) is satisfied: when \( X_w \) is rationally smooth.
- What do we do is purely combinatorial, since
  - \( P_w \) can be defined purely combinatorially and,
  - the geometric condition above can be described as pattern avoidance conditions for \( w \in W \).

2. \( P_w \) : the Poincaré Polynomial coming from \( X_w \) : the Schubert variety

- (Not needed) Schubert variety \( X_w := \overline{W(w)} / B \).
- Lower Bruhat interval \([i, w] := \{ u \in W | u \leq w \} \).
- \( P_w \) : the rank generating function for \([i, w] \). Here the rank is the number of inversions.
- (Carrell-Peterson Criteria) \( X_w \) is rationally smooth iff \( P_w \) is palindromic (Symmetric coefficients).
- By definition, we have \( P_{w^{-1}} = P_w \).

3. \( R_w \) coming from \( A_w \) : the inversion hyperplane arrangement

- \( A_w \) is the collection of hyperplanes \( \alpha(x) = 0 \) where \( \alpha > 0 \) s.t. \( w(\alpha) < 0 \).
- This is the generalization of the Coxeter arrangement, where we only take hyperplanes corresponding to inversions of \( w \).
- We define the distance enumerator polynomial \( R_w \):
  - For regions \( r, r' \) of \( A_w \), \( d(r, r') \) : Minimal \# of hyperplanes crossed to go from \( r \) to \( r' \).
  - \( R_\emptyset \) : Region containing the fundamental chamber.
  - \( R_w \) : the rank generating function for regions of \( A_w \). Here the rank is the distance from \( R_\emptyset \).
  - \( R_w \) is always palindromic.
  - \( R_{w^{-1}} = R_w \).

4. Sketch of the Main idea of the proof

- (Billey-Postnikov) When \( w \in W \) is rationally smooth, either \( w \) or \( w^{-1} \) decomposes as \( uv \) where \( u \in W_J, v \in W_I \) such that
  - \( u \) is maximal element of \( W_J \) below \( w \) (or \( w^{-1} \)),
  - \( J \) corresponds to leaf-removed subset of Dynkin diagram of \( W \).
- Then we get a factorization : \( P_{w^{-1}} = P_u P_v \).

**Key idea**

\( w \) has palindromic lower interval in \( W \) if and only if the interval is isomorphic to some maximal parabolic quotient of some Weyl group.

- Using this, we can decompose further : \( w = u_1 u_2 u_3 \) so that \( u_3 \) is the longest element of \( W_{I(J)} \) and \( I \) is the set of simple reflections in \( v \).
- Then \( A_w \) divides \( A_1 \) nicely, and we get \( R_w / R_1 = R_{w_1 w_2} / R_1 \).
- Next we show that \( R_{w_1 w_2} / R_1 = P_{w_1 w_2} \).
- Hence \( P_w \) and \( R_w \) factorizes similarly if \( w \) is rationally smooth.

5. Conclusion and further remarks

**Main theorem**

\( P_w = R_w \) if and only if \( X_w \) is rationally smooth.

- It would be interesting to check if our statement is also true for Coxeter groups in general.
- Whenever \( P_w = R_w \), the \( q \)-factors of \( P_w \) are exactly the roots of the characteristic polynomial of \( A_w \). Is it true for non-rationally smooth \( w \)?
- Is there a bijective proof for our statement?