Unitary Matrix Integrals, JM Elements, Primitive Factorizations

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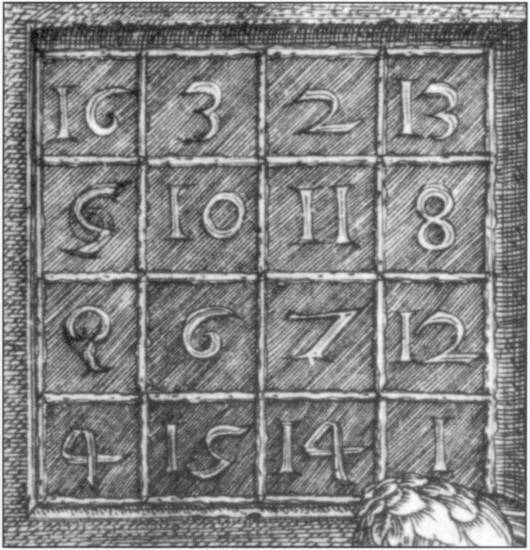
FPSAC 2010

August 3, 2010

Melencolia I, Albrecht Dürer, 1514

Melencolia detail: Dürer's magic square



Enumeration of magic squares

$H(m, n) :=$ number of $n \times n$ magic squares with magic sum m.

$$\boxed{H(m, n) = ?}$$

Some specific formulas

$$H(m,1) = 1 \qquad \rightsquigarrow \begin{bmatrix} m \end{bmatrix}$$

$$H(m,2) = m+1 \qquad \rightsquigarrow \begin{bmatrix} j & m-j \\ m-j & j \end{bmatrix}$$

$$H(m,3) = \binom{m+2}{4} + \binom{m+3}{4} + \binom{m+4}{4}$$

Some specific formulas

$H(1, n) = n!$ $\rightsquigarrow$ permutation matrices

$$H(2, n) = n! \left[\frac{z^n}{n!} \right] \frac{e^{z/2}}{\sqrt{1 - z}}$$

A general formula

Theorem (Diaconis-Gamburd)

For any $N \geq mn$,

$$\boxed{H(m,n) = \int_{\mathbf{U}(N)} e_m(U)^n \overline{e_m(U)}^n \, dU.}$$

$$\mathbf{U}(N) = \{N \times N \text{ complex matrices}, U^* = U^{-1}\}$$

$$dU = \text{normalized Haar measure on } \mathbf{U}(N)$$

$$\det(xI - U) = \sum_{m=0}^{N} (-1)^m e_m(U) x^{N-m}.$$

Why?

$$\boxed{e_m(U) = m\text{th elementary symmetric function of eigenvalues of } U}.$$

$$\Lambda = \underbrace{\bigoplus_{n=0}^{N} \Lambda^n}_{\text{isometric } \hookrightarrow L^2(\mathbf{U}(N))} \oplus \underbrace{\bigoplus_{n=N+1}^{\infty} \Lambda^n}_{\text{distorted}}.$$

$$\langle f, g \rangle_{\text{Hall}} = \int_{\mathbf{U}(N)} f(\epsilon_1(u), \dots, \epsilon_n(U), 0, \dots) \overline{g(\epsilon_1(u), \dots, \epsilon_n(U), 0, \dots)} dU$$

$$\langle e_\lambda, e_\mu \rangle = \langle h_\lambda, h_\mu \rangle = N_{\lambda\mu}.$$

Another point of view

$$\boxed{e_m(U) = \sum m \times m \text{ principal minors of } U}$$

$$e_2 \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{bmatrix} = \begin{vmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{vmatrix} + \begin{vmatrix} u_{11} & u_{13} \\ u_{31} & u_{23} \end{vmatrix} + \begin{vmatrix} u_{22} & u_{23} \\ u_{32} & u_{33} \end{vmatrix}$$

$$= u_{11}u_{22} - u_{12}u_{21} + u_{11}u_{33} - u_{13}u_{31}$$

$$+ u_{22}u_{33} - u_{23}u_{32}.$$

Monomial integrals:

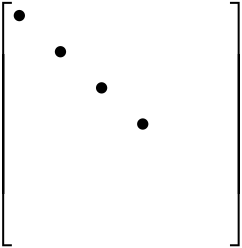
$$\langle u_{IJ}, u_{I'J'} \rangle_{L^2} = \int_{U(N)} u_{i(1)j(1)} \cdots u_{i(n)j(n)} \overline{u_{i'(1)j'(1)} \cdots u_{i'(n)j'(n)}} dU.$$

A random matrix

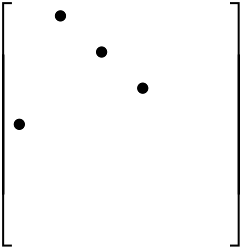
$$\begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} & u_{15} & \cdots & u_{1N} \\ u_{21} & u_{22} & u_{23} & u_{24} & u_{25} & \cdots & u_{2N} \\ u_{31} & u_{32} & u_{33} & u_{34} & u_{35} & \cdots & u_{3N} \\ u_{41} & u_{42} & u_{43} & u_{44} & u_{45} & \cdots & u_{4N} \\ u_{51} & u_{52} & u_{53} & u_{54} & u_{55} & \cdots & u_{5N} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{N1} & u_{N2} & u_{N3} & u_{N4} & u_{N5} & \cdots & u_{NN} \end{bmatrix}$$

{columns} is an orthonormal basis of $\mathbb{C}^N$.

Permutation correlators



$$\rightsquigarrow \int_{\mathbf{U}(N)} u_{11} u_{22} u_{33} u_{44} \overline{u_{11} u_{22} u_{33} u_{44}} dU$$



$$\rightsquigarrow \int_{\mathbf{U}(N)} u_{11} u_{22} u_{33} u_{44} \overline{u_{12} u_{23} u_{34} u_{41}} dU$$

4-point identity correlator: heuristics

$$\begin{bmatrix} \bullet & & & & & \\ & \bullet & & & & \\ & & \bullet & & & \\ & & & \bullet & & \\ & & & & & \\ & & & & & \end{bmatrix} \rightsquigarrow \int_{\mathbf{U}(N)} u_{11}u_{22}u_{33}u_{44}\overline{u_{11}u_{22}u_{33}u_{44}}\,dU$$

$$\begin{aligned} &\int_{\mathbf{U}(N)} u_{11}u_{22}u_{33}u_{44}\overline{u_{11}u_{22}u_{33}u_{44}}\,dU \\ &\sim \int_{\mathbf{U}(N)} |u_{11}|^2 dU \cdot \int_{\mathbf{U}(N)} |u_{22}|^2 dU \cdot \int_{\mathbf{U}(N)} |u_{33}|^2 dU \cdot \int_{\mathbf{U}(N)} |u_{44}|^2 dU \\ &\sim \frac{1}{N} \cdot \frac{1}{N} \cdot \frac{1}{N} \cdot \frac{1}{N} \end{aligned}$$

4-point identity correlator: true value

4-point identity correlator, $\pi = (1)(2)(3)(4)$:

$$\int_{\mathbf{U}(N)} u_{11} u_{22} u_{33} u_{44} \overline{u_{11} u_{22} u_{33} u_{44}} dU$$
$$= \frac{1}{N^4} + \frac{6}{N^6} + \frac{41}{N^8} + \frac{316}{N^{10}} + \frac{2631}{N^{12}} + \frac{22826}{N^{14}} + \frac{202021}{N^{16}} + \dots$$

Perturbative expansion $\rightsquigarrow$ generating function

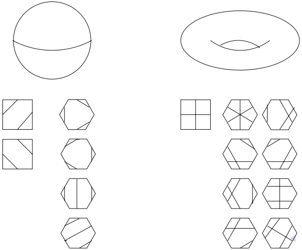
Genus expansion: Hermitian matrix models

Theorem (Harer-Zagier)

$$\int_{\mathbf{H}(N)} \mathrm{tr}(H^{2n}) GUE(dH) = \sum_{g\geq 0} \frac{\varepsilon_g(n)}{N^{2g}},$$

where

$$\varepsilon_g(n) = \#\text{one-face maps with } n \text{ edges on genus } g \text{ surface.}$$



Jucys-Murphy elements

$$C_{21111} = C_1(6) = \begin{bmatrix} (1\ 2) & (1\ 3) & (1\ 4) & (1\ 5) & (1\ 6) \\ 0 & (2\ 3) & (2\ 4) & (2\ 5) & (2\ 6) \\ 0 & 0 & (3\ 4) & (3\ 5) & (3\ 6) \\ 0 & 0 & 0 & (4\ 5) & (4\ 6) \\ 0 & 0 & 0 & 0 & (5\ 6) \end{bmatrix}$$

$$\begin{aligned} J_2 &= (1\ 2) \\ J_3 &= (1\ 3) + (2\ 3) \\ J_4 &= (1\ 4) + (2\ 4) + (3\ 4) \\ J_5 &= (1\ 5) + (2\ 5) + (3\ 5) + (4\ 5) \\ J_6 &= (1\ 6) + (2\ 6) + (3\ 6) + (4\ 6) + (5\ 6) \end{aligned}$$

JM elements commute, but

$$\{J_1, J_2, \ldots, J_n\} \not\subset \mathcal{Z}(n).$$

The JM specialization

Theorem (Jucys)

We have a specialization $\Xi_n : \Lambda \to \mathcal{Z}(n)$ *defined by*

$$f(\Xi_n) = f(J_1, J_2, \dots, J_n, 0, 0, \dots).$$

Proof.

$$\begin{aligned} e_k(\Xi_n) &= \sum_{\pi \in \mathbf{S}(n)} \#\{\pi = (s_1\ t_1)(s_2\ t_2)\dots(s_k\ t_k)\ :\ t_1 < t_2 < \dots < t_k\}\pi \\ &= \sum_{|\pi|=k} \pi \\ &= \sum_{|\mu|=k} C_\mu(n) \in \mathcal{Z}(n). \end{aligned}$$

Class expansion problem

$$f(\Xi_n) = \sum_{\mu} a^f_\mu(n) C_\mu(n)$$

$$\boxed{a^f_\mu(n) = \ ?}$$

Class expansion: examples

$$e_4(\Xi_n) = 1\,C_4(n) + 1\,C_{31}(n) + 1\,C_{22}(n) + 1\,C_{1111}(n)$$

$$p_4(\Xi_n) = 1\,C_4(n) + (3n-4)C_2(n) + 4\,C_{11}(n) + \frac{1}{6}n(n-1)(4n-5)C_0(n)$$

$$\begin{aligned} h_4(\Xi_n) = {} & 14\,C_4(n) + 5\,C_{31}(n) + 4\,C_{22}(n) + 2\,C_{211}(n) + 1\,C_{1111}(n) \\ & + (n^2+8n-23)C_2(n) + \frac{1}{2}(n^2+7n-4)C_{11}(n) \\ & + \frac{1}{24}n(n-1)(3n^2+17n-34)C_0(n) \end{aligned}$$

The connection with permutation correlators

Theorem

For any $\pi \in C_\mu(n)$,

$$\int_{\mathbf{U}(N)} u_{11} \dots u_{nn} \overline{u_{1\pi(1)} \dots u_{n\pi(n)}} dU = \sum_{k \geq 0} \frac{(-1)^k a_\mu^{h_k}(n)}{N^k}.$$

Connection coefficients

$$C_\alpha(n)C_\beta(n)\ldots C_\zeta(n) = \sum_\mu b_\mu^{\alpha\beta\ldots\zeta}(n)C_\mu(n)$$

$$C_2(n)C_{11}(n) = 5C_4(n) + 4C_{31}(n) + 1C_{211}(n) + 3(n-3)C_2(n) + 4(n-4)C_{11}(n).$$

Top connection coefficients

Rumour has it there is an explicit combinatorial formula for all top connection coefficients.

Top class coefficients

$$m_{31}(\Xi_n) = 4C_4(n) + 1C_{31}(n) + 2(3n-7)C_2(n) + 2(2n-3)C_{11}(n) + \frac{1}{3}n(n-1)(n-2)C_0(n)$$

$$m_{22}(\Xi_n) = 2C_4(n) + 1C_{22}(n) + \frac{1}{2}(n^2-n-4)C_2(n) + 2C_{11}(n) + \frac{1}{24}n(n-1)(n-2)(3n-1)C_0(n)$$

$$m_{211}(\Xi_n) = 6C_4(n) + 3C_{31}(n) + 2C_{22}(n) + 1C_{211}(n) + \frac{1}{2}(n-3)(n+2)C_2(n) + \frac{1}{2}(n^2-n-4)C_{11}(n)$$

Top class coefficients

Theorem

For $|\mu| = |\lambda|$ *we have*

$$a_{\mu}^{m_\lambda} = \sum_{(\lambda^1,\ldots,\lambda^{\ell(\mu)})\in\mathcal{R}(\lambda,\mu)} \mathrm{RC}(\lambda^1)\ldots\mathrm{RC}(\lambda^{\ell(\mu)}),$$

where

$$\mathcal{R}(\lambda,\mu) = \{(\lambda^1,\ldots,\lambda^{\ell(\mu)}) : \lambda^i \vdash \mu_i,\ \lambda^1\cup\cdots\cup\lambda^{\ell(\mu)} = \lambda\}$$

and

$$\mathrm{RC}(\lambda) = \frac{|\lambda|!}{(|\lambda|-\ell(\lambda)+1)!\prod m_i(\lambda)!}$$

are refined Catalan numbers:

$$\sum_{\lambda\vdash k}\mathrm{RC}(\lambda) = \mathrm{Cat}_k\,.$$

Class coefficients: combinatorial approach

Proof.
$a_\mu^{m_\lambda}$ counts minimal primitive factorizations of $\pi \in C_\mu(n)$ with frequencies prescribed by λ :

$$\pi = \underbrace{(*2)\dots(*2)}\underbrace{(*3)\dots(*3)}\dots\underbrace{(*n)\dots(*n)}.$$

Example of a factorization counted by $a_9^{m_{33111}}$:

$$(1\ 2\ \dots\ 10) = \underbrace{(2\ 3)}\underbrace{(4\ 5)(3\ 5)(1\ 5)}\underbrace{(7\ 8)(6\ 8)(5\ 8)}\underbrace{(8\ 9)}\underbrace{(9\ 10)}.$$

Left and right sequences:

$\mathcal{L} = 2\ 4\ 3\ 1\ 7\ 6\ 5\ 8\ 9$ (312-avoiding perm of type λ)
$\mathcal{R} = 3\ 5\ 5\ 5\ 8\ 8\ 8\ 9\ 10$ (primitive (reverse) parking fcn of type λ).

□

Class coefficients: combinatorial approach

Corollary

$$a_{\mu}^{h_{|\mu|}} = \prod_{i=1}^{\ell(\mu)} \mathrm{Cat}_{\mu_i} .$$

Example

$$\begin{aligned} h_4(\Xi_n) &= \mathrm{Cat}_4\, C_4(n) + \mathrm{Cat}_3\, \mathrm{Cat}_1\, C_{31}(n) + \mathrm{Cat}_2\, \mathrm{Cat}_2\, C_{22}(n) \\ &+ \mathrm{Cat}_2\, \mathrm{Cat}_1\, \mathrm{Cat}_1\, C_{211}(n) + \mathrm{Cat}_1\, \mathrm{Cat}_1\, \mathrm{Cat}_1\, \mathrm{Cat}_1\, C_{1111}(n) \\ &+ (n^2 + 8n - 23) C_2(n) + \frac{1}{2}(n^2 + 7n - 4) C_{11}(n) \\ &+ \frac{1}{24} n(n-1)(3n^2 + 17n - 34) C_0(n) \end{aligned}$$

Top class coefficients in action

Corollary

For any $\pi \in C_\mu(n)$,

$$(-1)^{|\mu|} N^{n+|\mu|} \int_{\mathbf{U}(N)} u_{11} \dots u_{nn} \overline{u_{1\pi(1)} \dots u_{n\pi(n)}} dU = \prod_{i=1}^{\ell(\mu)} \mathrm{Cat}_{\mu_i} + O\left(\frac{1}{N^2}\right).$$

Theorem

Random matrix $U = [u_{ij}] \in \mathbf{U}(N)$. *Random measure*

$$\mu_N = \frac{1}{N^2} \sum_{i,j} \delta_{u_{ij}} \in \mathcal{M}(\mathbb{D}).$$

Deterministic limit:

$$\mu_N \to \delta_0$$

weakly in $\mathcal{M}(\mathbb{D})$.

Macdonald's symmetric functions

Theorem (Macdonald)

There exists a basis $\{g_\mu\}$ *of* Λ *such that*

$$g_\alpha g_\beta \ldots g_\zeta = \sum_\mu b_\mu^{\alpha\beta\ldots\zeta} g_\mu$$

Theorem

Forgotten symmetric functions $\longrightarrow$ *Macdonald symmetric functions:*

$$(-1)^{|\lambda|} f_\lambda = \sum_{\mu \vdash |\lambda|} a_\mu^{m_\lambda} g_\mu.$$

Connection coefficients: algebraic approach

Theorem (Frobenius-Burnside)

$$b_{\mu}^{\alpha\beta\ldots\zeta}(n) = \sum_{\lambda \vdash n} \omega_{\mu}(\lambda)\omega_{\alpha}(\lambda)\omega_{\beta}(\lambda)\ldots\omega_{\zeta}(\lambda)\frac{(\dim\lambda)^2}{n!}$$
$$= \langle \omega_{\mu}(\lambda)\omega_{\alpha}(\lambda)\omega_{\beta}(\lambda)\ldots\omega_{\zeta}(\lambda)\rangle_{\mathrm{Plancherel}(n)},$$

where $\omega_{\mu}(\lambda) = |C_{\mu}(n)|\frac{\chi^{\lambda}(\pi)}{\dim\lambda}, \;\; \pi \in C_{\mu}(n).$

Connection coefficients: algebraic approach

$$\omega_1(\lambda) = \sum_{\square \in \lambda} c(\square)$$

$$\omega_{n-1}(\lambda) = \pm[\lambda \text{ is a hook}]$$

Theorem (Jackson, Shapiro-Shapiro-Vainshtein)

The number of factorizations of $(1\ 2\ \ldots\ n)$ *into* $n-1+2g$ *transpositions is*

$$b_{n-1}^{\overbrace{11\ldots1}^{n-1+2g}}(n) = n^{n-2} n^{2g} \binom{n-1+2g}{n-1} \left[\frac{z^{2g}}{(2g)!}\right] \left(\frac{\sinh z/2}{z/2}\right)^{n-1}$$

Class coefficients: algebraic approach

Theorem (Analogue Frobenius-Burnside (Jucys))

$$a^f_\mu(n) = \frac{1}{|C_\mu(n)|}\sum_{\lambda\vdash n} f(A_\lambda)\omega_\mu(\lambda)\frac{(\dim\lambda)^2}{n!} = \sum_{\lambda\vdash n}\frac{f(A_\lambda)}{H_\lambda}\chi^\lambda(\pi).$$

We can use this to:

- Obtain a formula for $\mathbf{U}(N)$-correlators in terms of $\mathbf{S}(n)$-characters.
- Obtain an analogue of the JSSV formula.

Class coefficients: algebraic approach

Theorem

Ordinary generating function:

$$\Phi_\mu(z;n) := \sum_{k\geq 0} a_\mu^{h_k}(n) z^k.$$

Then

$$\Phi_\mu(z;n) = \sum_{\lambda\vdash n} \frac{\chi^\lambda(\pi)}{\prod_{\Box\in\lambda}(1-c(\Box)z)}.$$

Corollary

$$\int_{\mathbf{U}(N)} u_{11}\dots u_{nn}\overline{u_{1\pi(1)}\dots u_{n\pi(n)}} = \sum_{\lambda\vdash n} \frac{\chi^\lambda(\pi)}{\prod_{\Box\in\lambda}(N+c(\Box))}.$$

Class coefficients: analogue JSSV formula

Theorem

Ordinary generating function:

$$\Phi_{n-1}(z;n) = \frac{\mathrm{Cat}_{n-1}\, z^{n-1}}{(1-z^2)(1-4z^2)\ldots(1-(n-1)^2z^2)}.$$

Equivalently,

$$a^{h_k}_{n-1}(n) = \mathrm{Cat}_{n-1} \cdot T(n-1+g, n-1),$$

where $T(m,n)$ denotes the Carlitz-Riordan central factorial number. Equivalently,

$$a^{h_k}_{n-1}(n) = \mathrm{Cat}_{n-1} \binom{2n-2+2g}{2n-2} \left[\frac{z^{2g}}{(2g)!}\right] \left(\frac{\sinh z/2}{z/2}\right)^{2n-2}.$$

Central factorial numbers

Stirling numbers – specialization $h_g \mapsto h_g(1, 2, \dots, n, 0, 0, \dots)$:

$$\frac{z^n}{(1-z^2)(1-2z^2)\dots(1-nz^2)} = \sum_{g \geq 0} S(n+g, n) z^{n+2g}.$$

Central factorial numbers ("coloured Stirling numbers") — specialization $h_g \mapsto h_g(1^2, 2^2, \dots, n^2, 0, 0, \dots)$:

$$\frac{z^n}{(1-z^2)(1-4z^2)\dots(1-n^2z^2)} = \sum_{g \geq 0} T(n+g, n) z^{n+2g}.$$

Central factorial numbers

$T(3,2)$ enumerates certain 2-block partitions of $\{1,2,3,1,2,3\}$.

$$11 \sqcup 2233$$
$$1122 \sqcup 33$$
$$1133 \sqcup 22$$
$$113 \sqcup 223$$
$$113 \sqcup 223$$

New interpretations of central factorial numbers

- *Combinatorial:* $\mathrm{Cat}_{n-1} \cdot T(n-1+g, n-1)$ counts primitive factorizations of a cycle:

$$(1\ 2\ \ldots\ n) = (s_1\ t_1)(s_2\ t_2)\ldots(s_{n-1+2g}\ t_{n-1+2g})$$
$$2 \leq t_1 \leq \cdots \leq t_{n-1+2g} \leq n$$

- *Probabilistic:* fluctuation series for cyclic entry correlators in the Circular Unitary Ensemble:

$$(-1)^{n-1} N^{2n-1} \int_{\mathbf{U}(N)} u_{11} u_{22} \ldots u_{nn} \overline{u_{12} u_{23} \ldots u_{n1}} dU$$
$$= \mathrm{Cat}_{n-1} \sum_{g \geq 0} \frac{T(n-1+g, n-1)}{N^{2g}}.$$

Further work

- MATSUMOTO: $\mathbf{U}(N) \rightsquigarrow \mathbf{O}(N)$, class algebra in $\mathbb{C}\mathbf{S}(n) \rightsquigarrow$ double coset algebra in Gelfand pair $(\mathbf{S}(2n), \mathbf{H}(n))$.
- NOVAK: More on the unitary group coming soon.