Fully Packed Loop Configurations and Littlewood–Richardson coefficients

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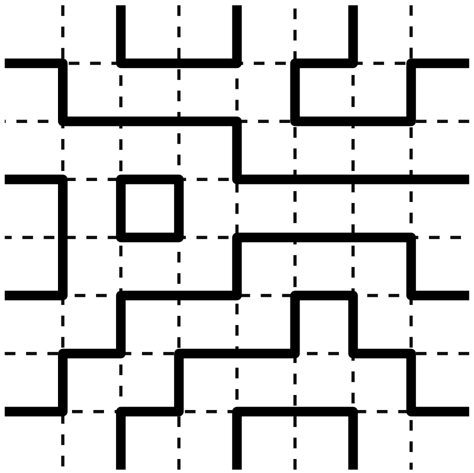
FPSAC 22, San Francisco, August 4th 2010.

Rough Outline

Fully Packed Loops in a square grid



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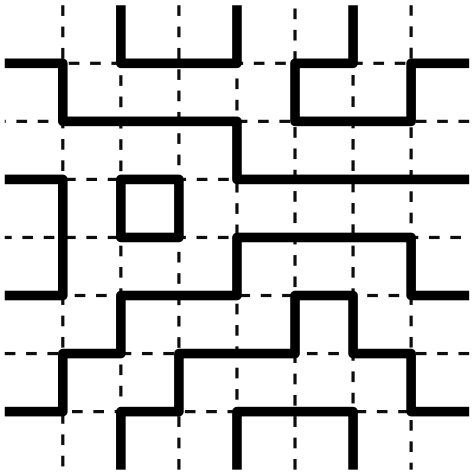

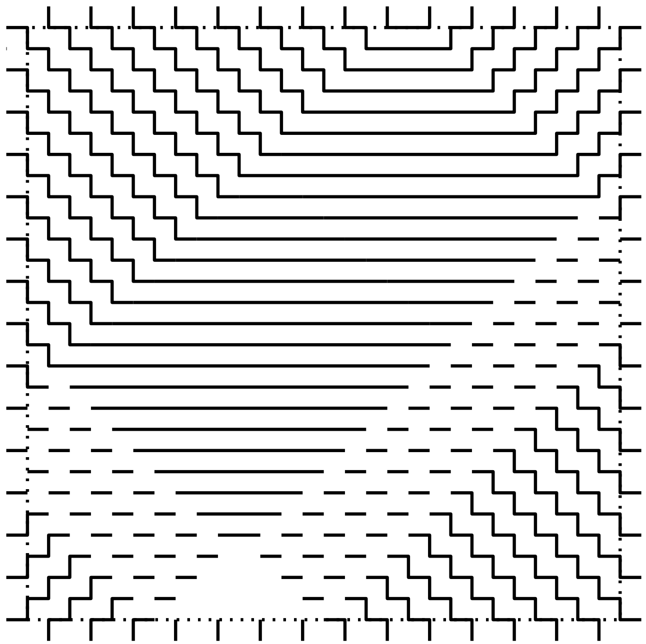

From the square to the triangle

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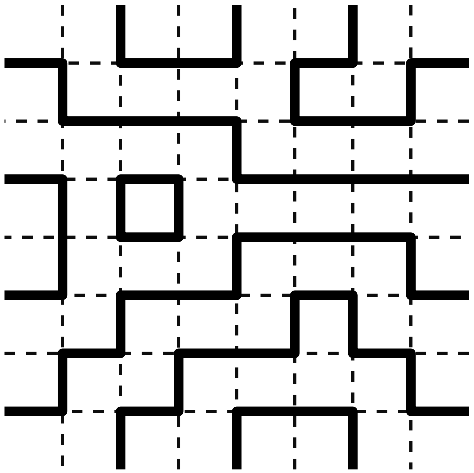

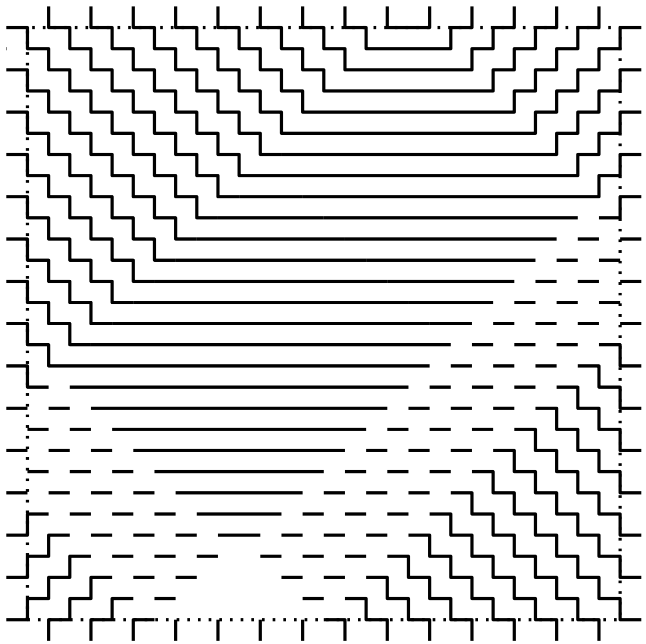

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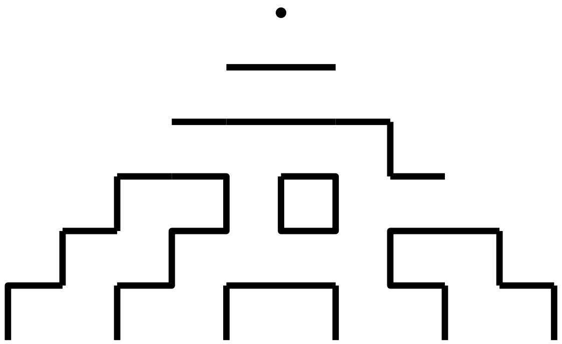
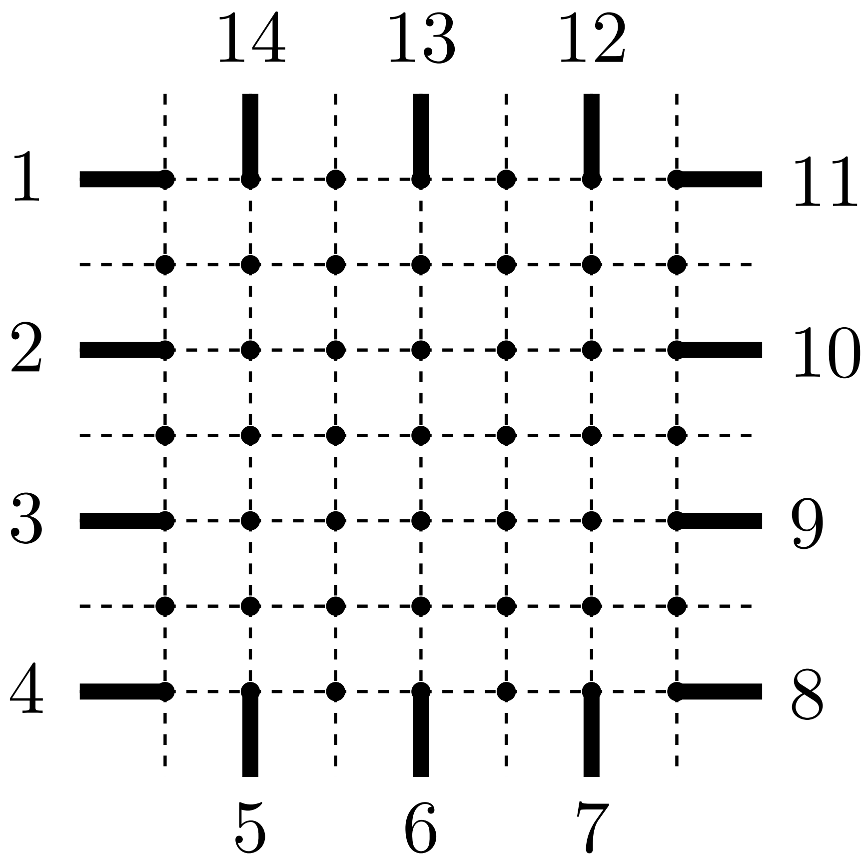
Fully Packed Loops in a triangle



FPL configurations : Definition

Start with the square grid G_n with n^2 vertices and $4n$ external edges, and pick every other edge on the boundary (starting with the topmost on the left).



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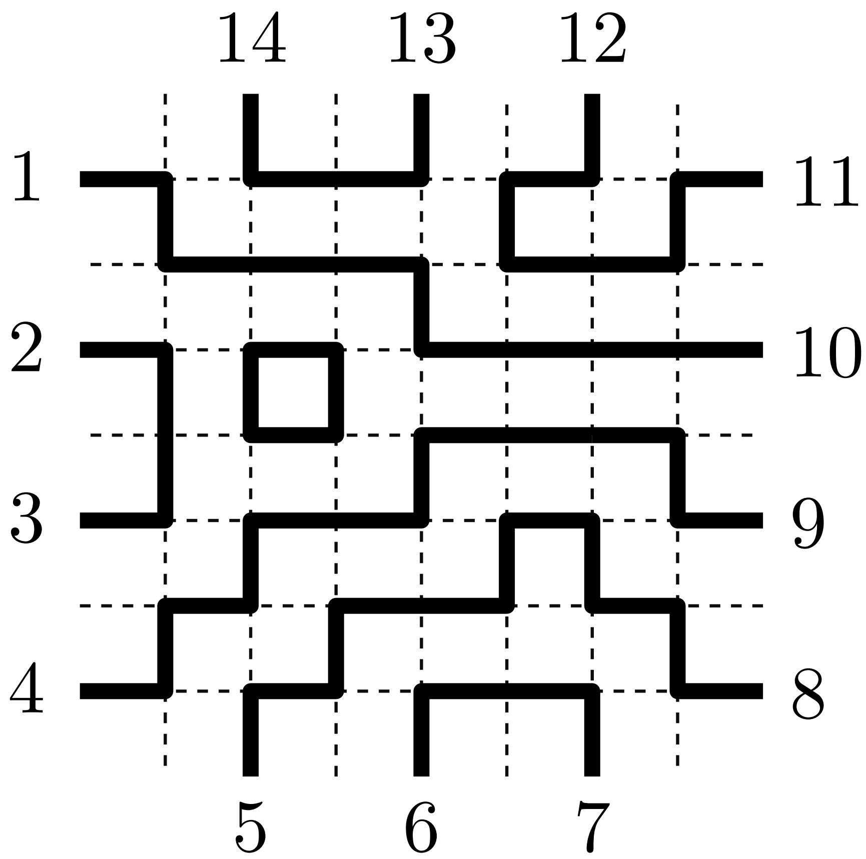
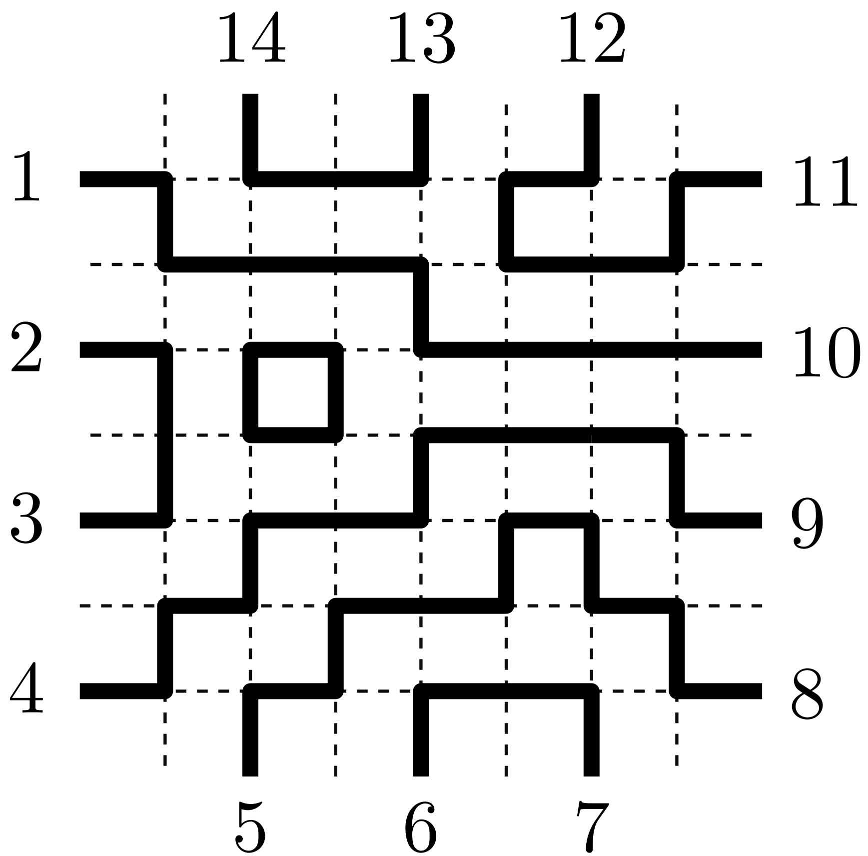


A Fully Packed Loop (FPL) configuration of size n is a subgraph of G_n with exactly 2 edges incident to each vertex.

FPL configurations : Enumeration

FPL configurations are in simple bijection with numerous objects : alternating sign matrices (ASMs), height matrices, configurations of the six vertex model, Gog triangles,...



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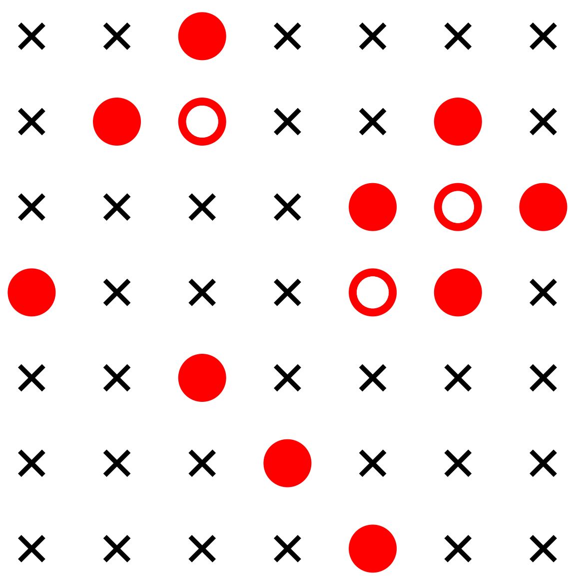

FPL of size n

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ASMs of size n

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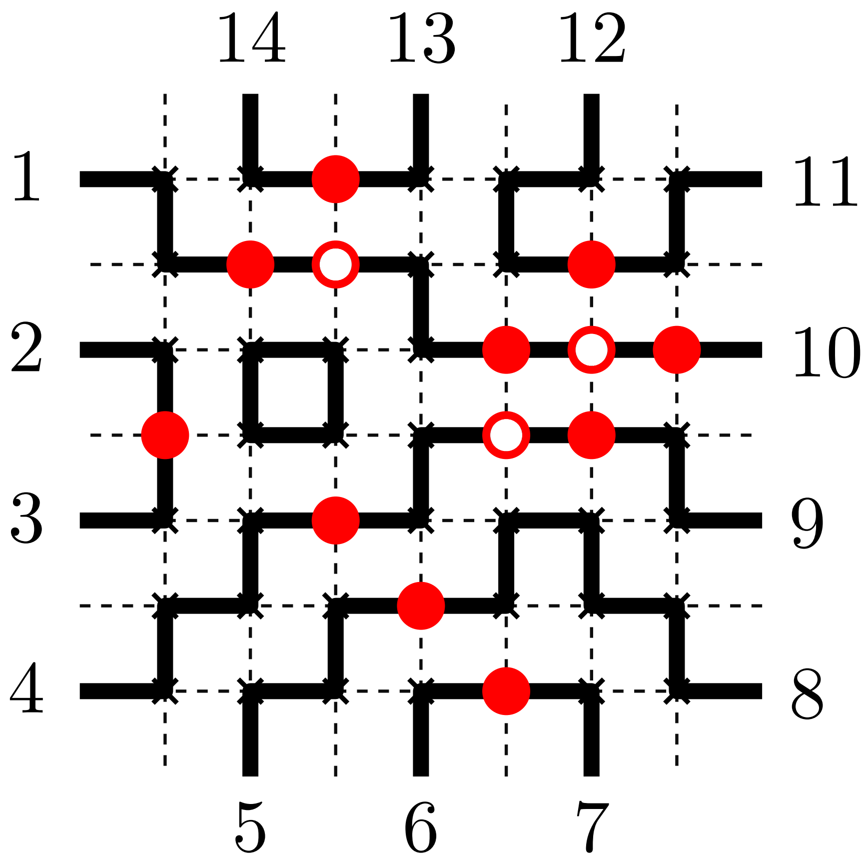

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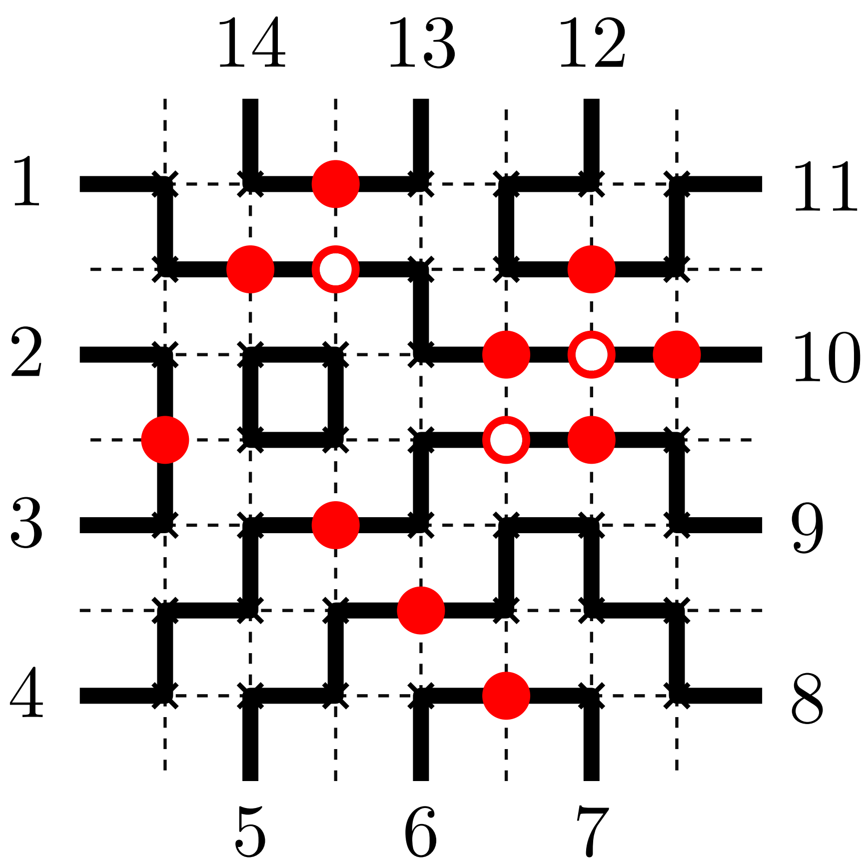

$$|FPL_n| = A_n = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

[Zeilberger '96, Kuperberg '96]

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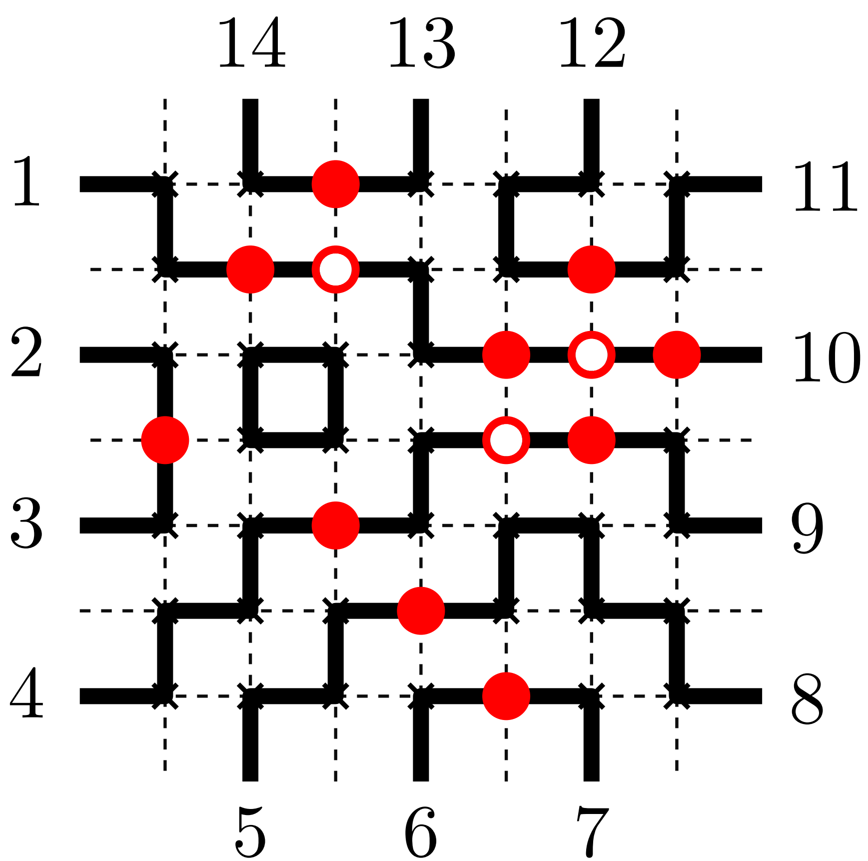

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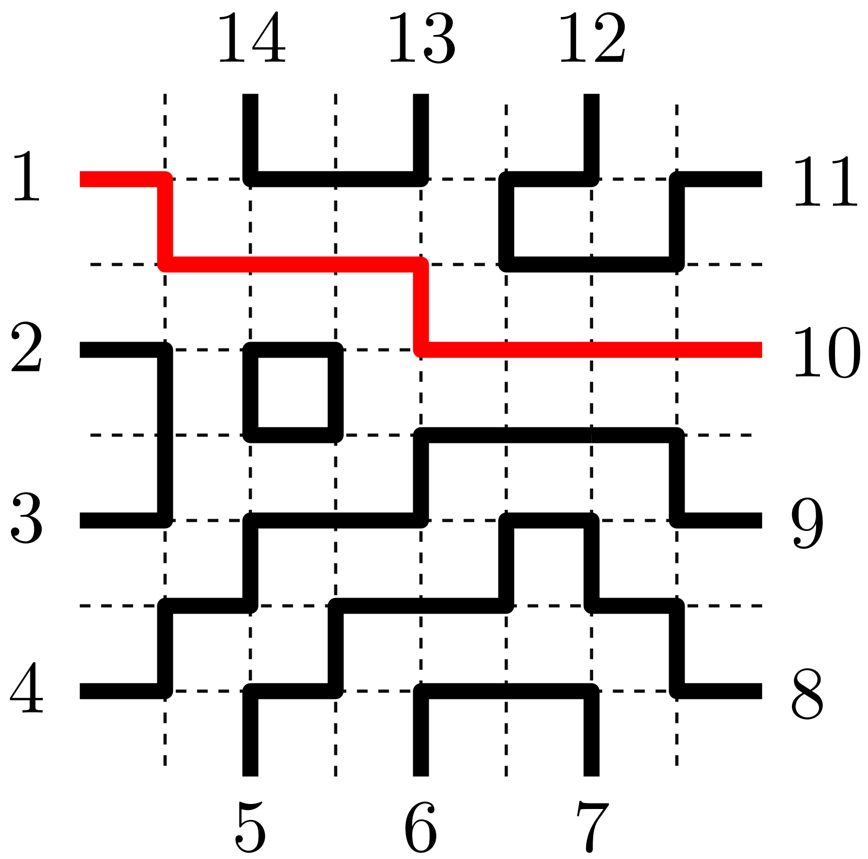
[Zeilberger '96, Kuperberg '96]

Why study FPLs rather than ASMs ?

FPL configurations : Refined enumeration

Every FPL configuration determines a link pattern on the external edges of the grid G_n, where link pattern = set of n noncrossing chords between $2n$ labeled points on a disk.



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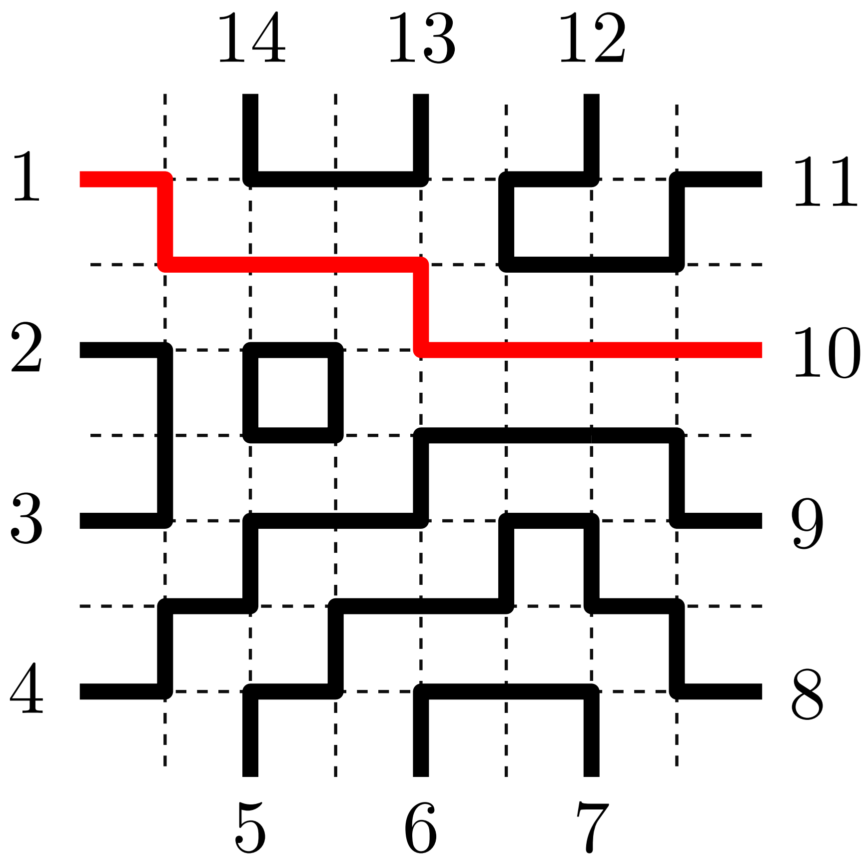
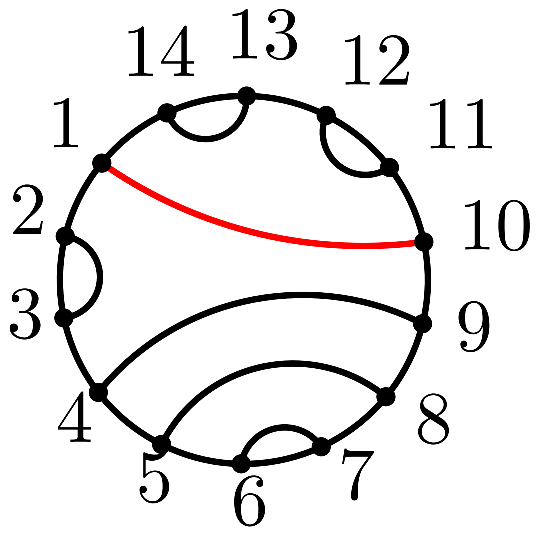


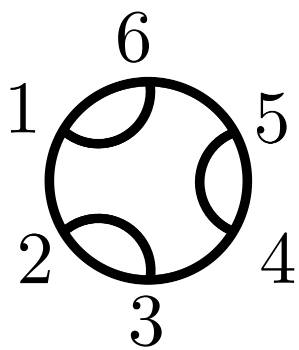


$$|LP_n| = C_n := \frac{1}{n+1}\binom{2n}{n}$$

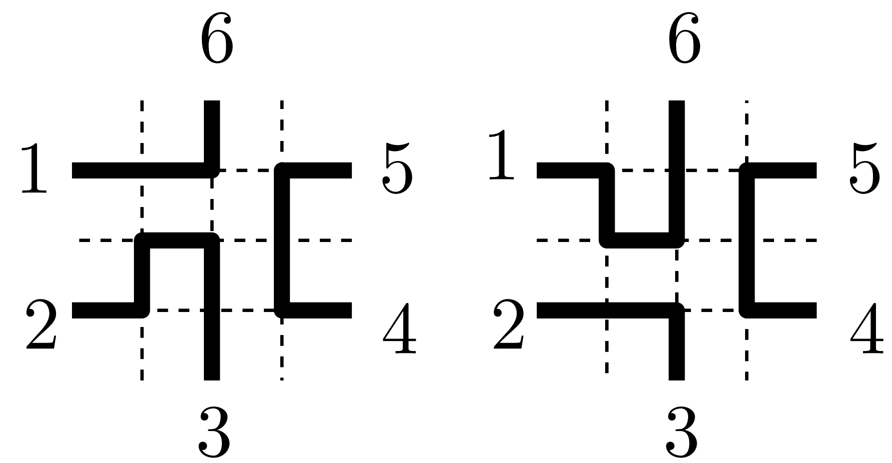
FPL configurations : Refined enumeration

Main problem : given a link pattern X, how many FPL configurations induce the link pattern X ?

We note A_X this number.

If $X =$ 

$A_X = 2$



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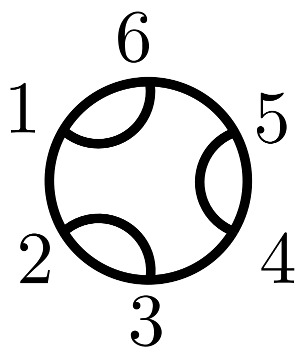
If $X =$ 

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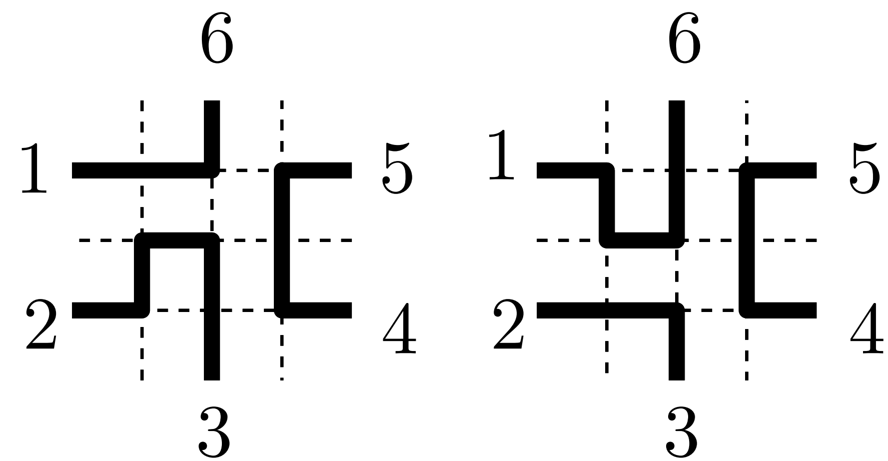
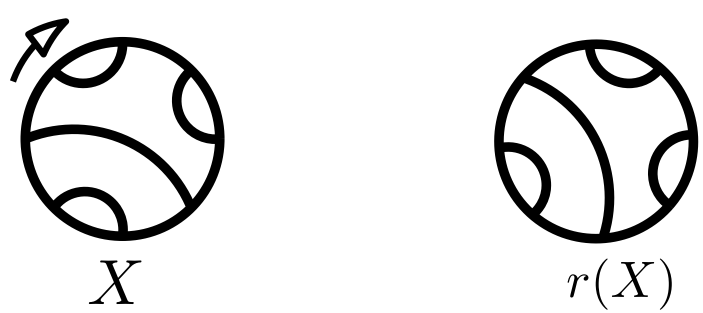
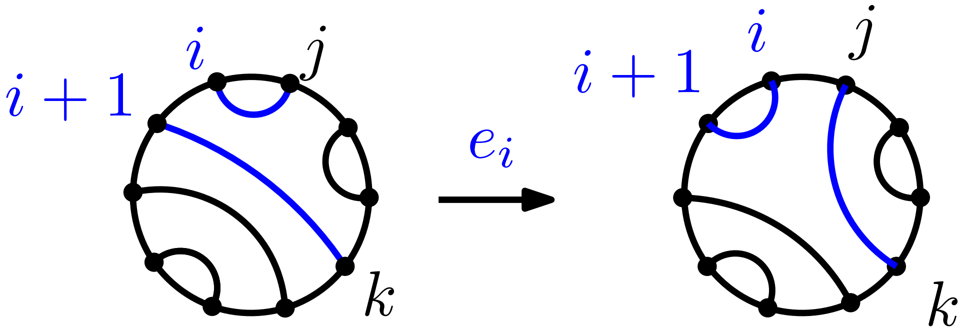


Wieland's rotation Given a link pattern X, consider the rotated pattern $r(X)$ obtained by $\{i, j\} \mapsto \{i+1, j+1\}$. Then $A_X = A_{r(X)}$.



The Razumov-Stroganov correspondence

For $i = 1 \ldots 2n$ let e_i act on link patterns by
$\{i, j\}, \{i+1, k\} \in X \to \{i, i+1\}, \{j, k\} \in e_i(X)$.



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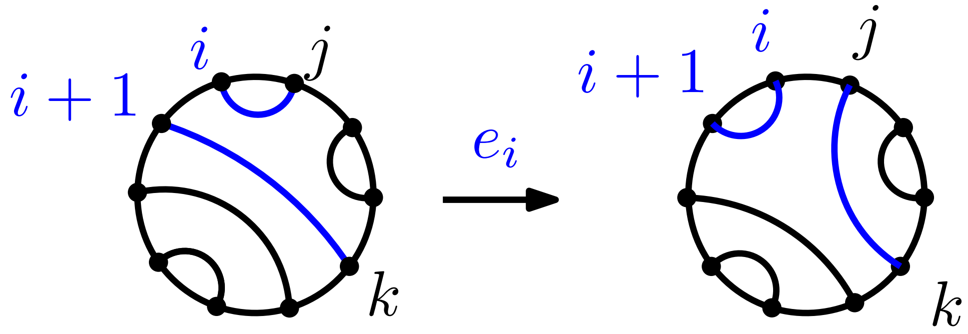

RS correspondence [RS '01, Cantini and Sportiello '10] :

$$\forall X, \quad 2n A_X = \sum_{(i,Y), e_i(Y)=X} A_Y$$

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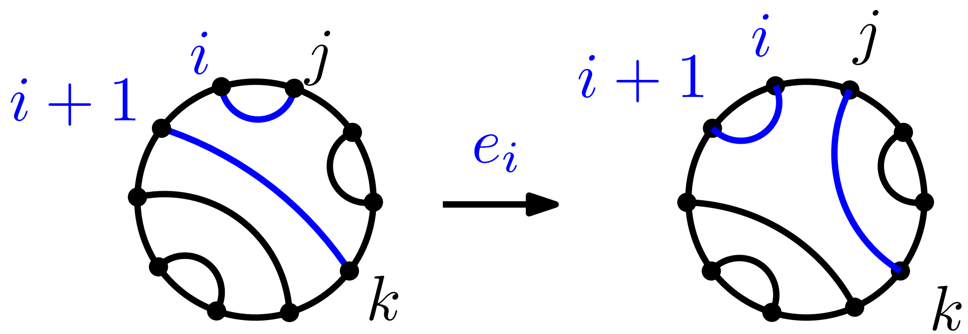
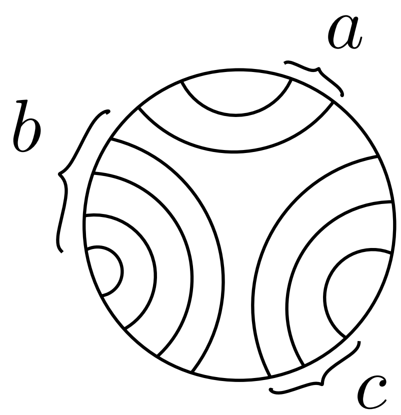


RS correspondence [RS '01, Cantini and Sportiello '10] :

$$\forall X, \quad 2n A_X = \sum_{(i,Y), e_i(Y)=X} A_Y$$

These relations completely characterize the A_X.
Di Francesco and Zinn Justin had previously obtained results on the solutions of these relations
$\to$ these are now applicable to the quantities A_X.

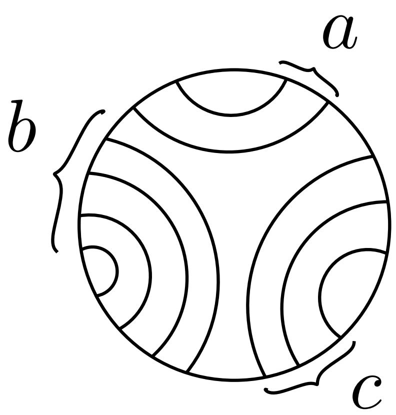
Link patterns X with nice expressions for A_X

$X =$ 

$$A_X = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$$

[Zinn-Justin, Zuber, Di Francesco]

Link patterns X with nice expressions for A_X

$X =$ 

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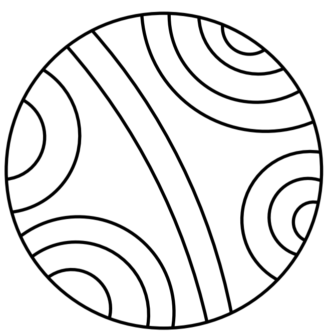
$X =$ 

$A_X =$ Complicated determinant formula

[Caselli and Krattenthaler '04, Zinn-Justin '08]

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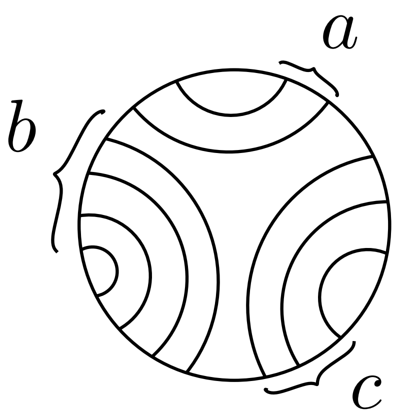
$X =$ 

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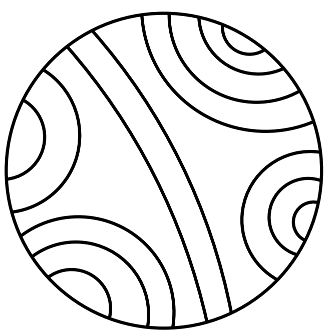
$X =$ 

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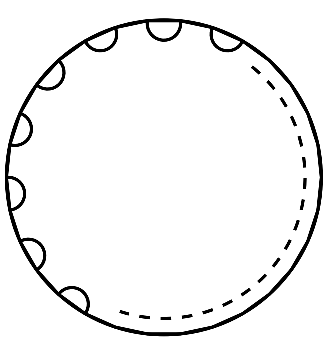
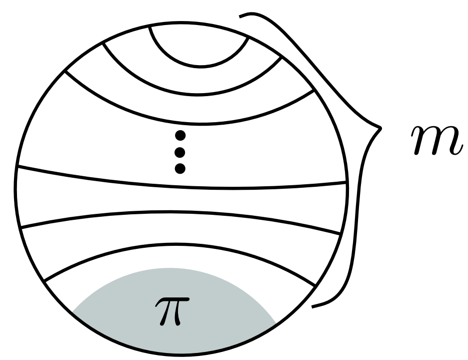
$X =$ 

$$A_X = A_{n-1}.$$

[Di Francesco and Z-J, Cantini and Sportiello]

Link patterns with nested arches

We consider now integers $n, m \geq 0$, and link patterns with m nested arches, and π is a noncrossing matching with n arches.

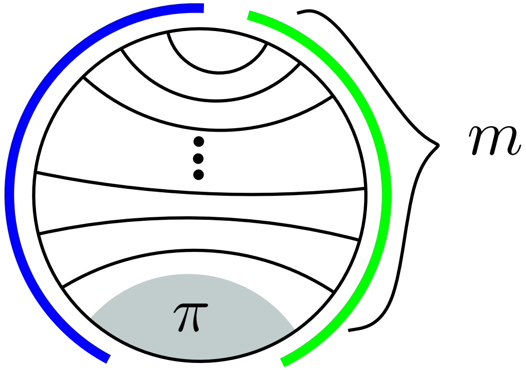


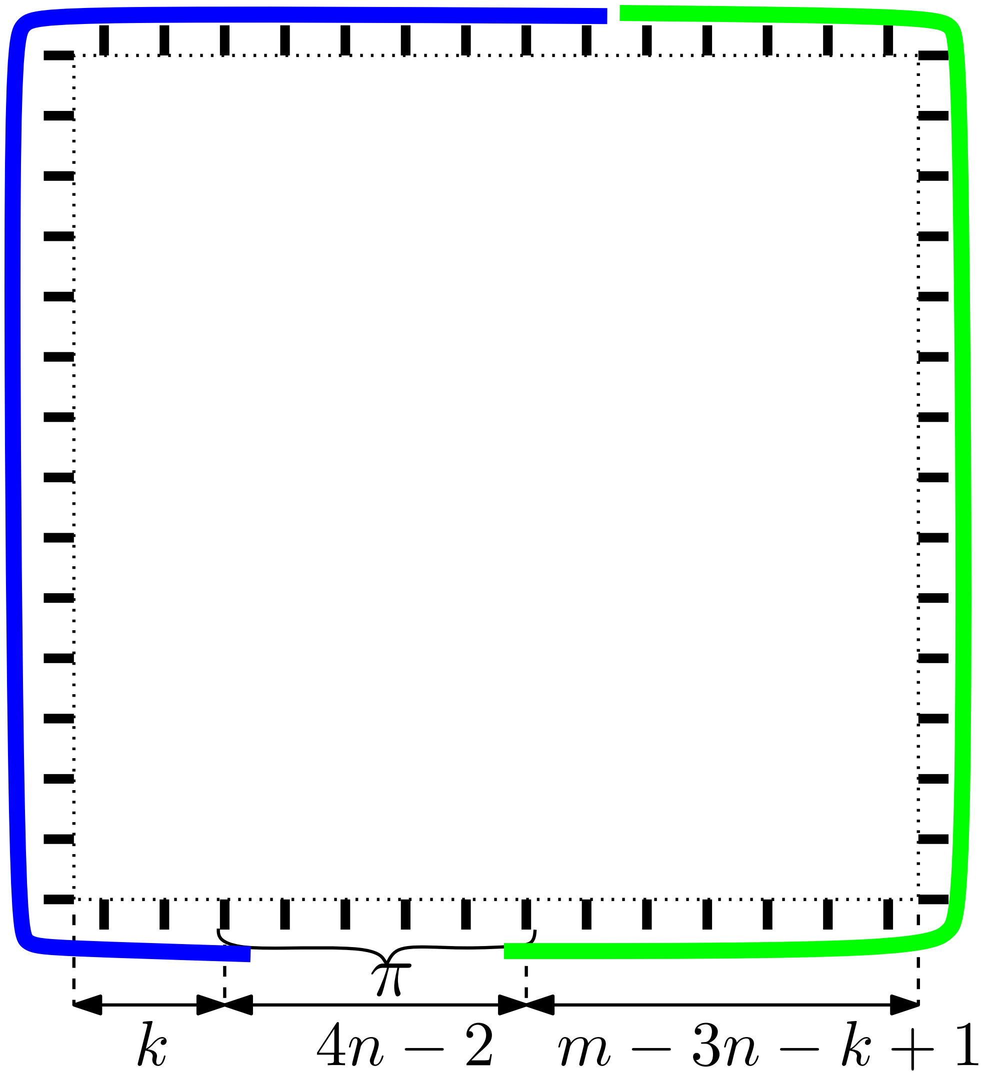
$$X = \pi \cup m$$

We will determine an expression for $A_{\pi \cup m}$, based on FPLs in a triangle. ($\to$ The case $m = 0$ gives the usual numbers A_X.)

Link patterns with nested arches

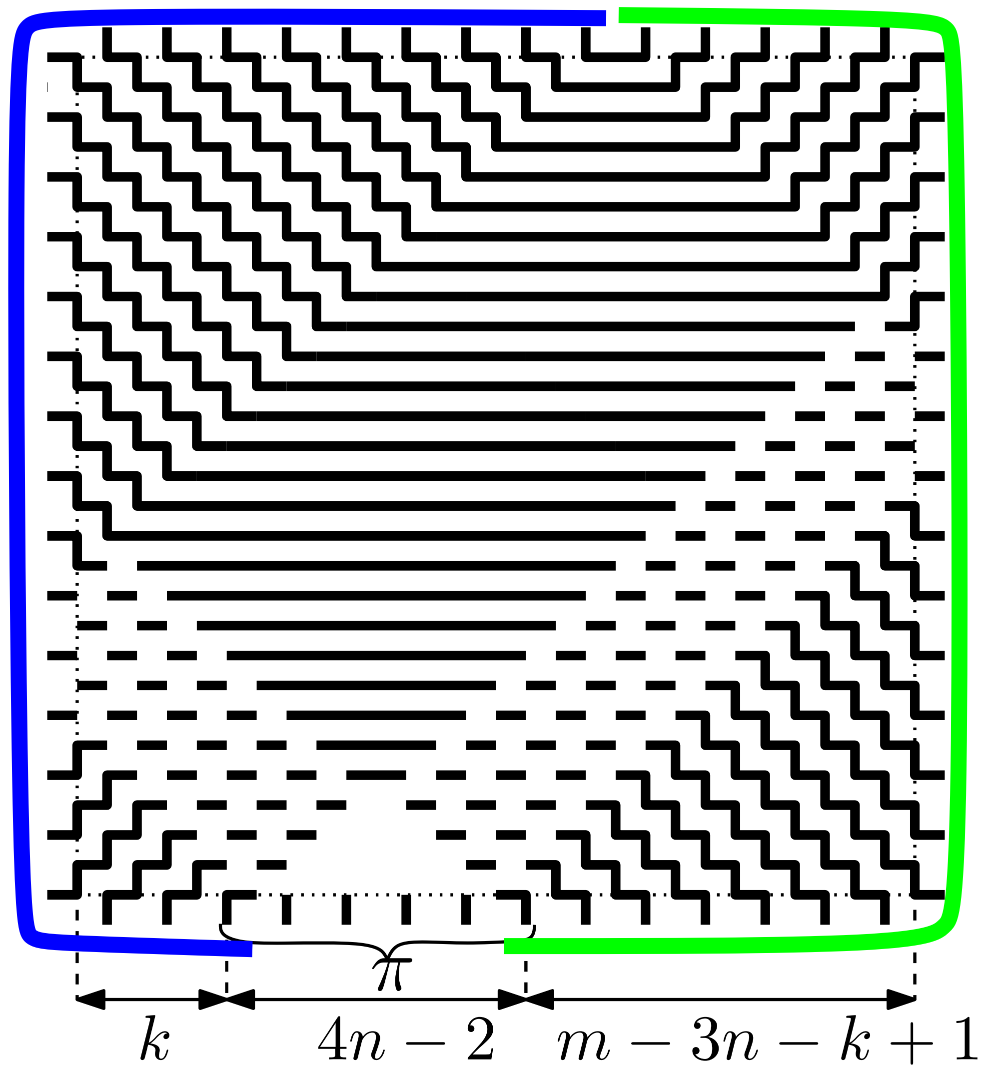
We suppose $m \geq 3n-1$, and choose k such that $0 \leq k \leq m-(3n-1)$.





Link patterns with nested arches

Fixed edges based on a lemma from [de Gier, '02].



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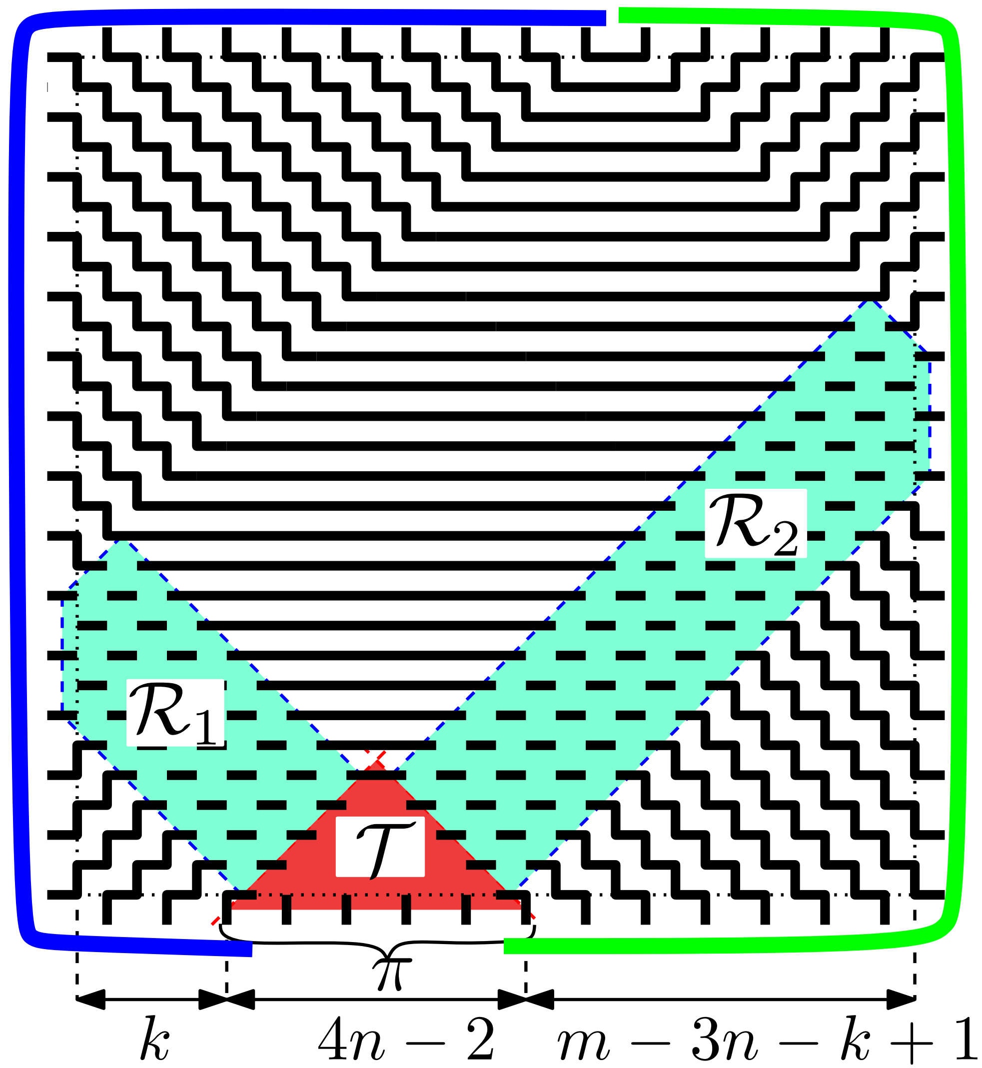
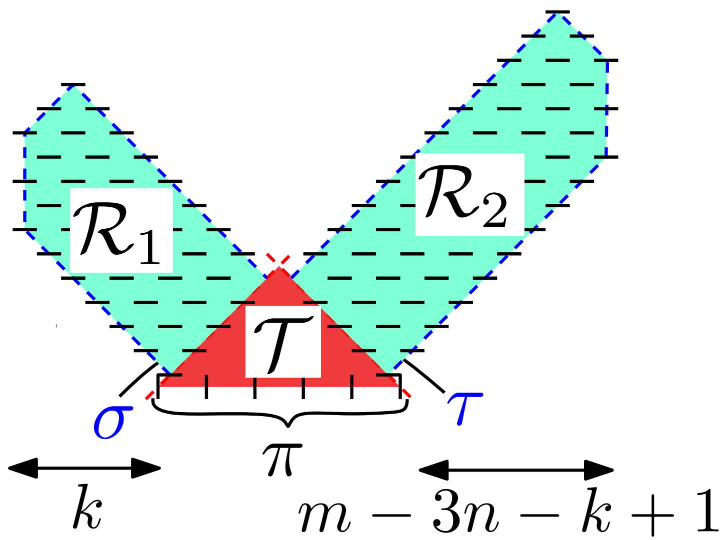
Three regions appear



Link patterns with nested arches

We can then write, for $m \geq 3n-1$ and $0 \leq k \leq m-(3n-1)$

$$A_{\pi \cup m} = \sum_{\sigma,\tau} |\mathcal{R}_1(\sigma, k)| \times t^{\pi}_{\sigma,\tau} \times |\mathcal{R}_2(\tau, m-3n-k+1)|$$



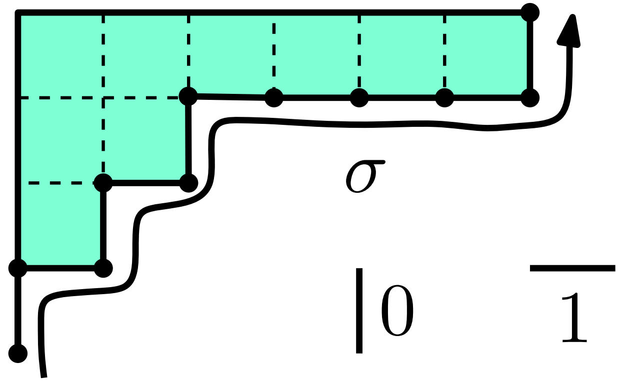
- σ, τ are words of length $2n$ on $\{0,1\}$;
- $\mathcal{R}_1(\sigma, .), \mathcal{R}_2(\tau, .)$ are the sets of FPLs in $\mathcal{R}_1$ and $\mathcal{R}_2$.
- $t^{\pi}_{\sigma,\tau}$ is the number of FPL configurations in the triangle $\mathcal{T}$.

Words and Shapes

Let $\sigma = \sigma_1 \ldots \sigma_p$ be a word in $\{0,1\}^p$; we write $|\sigma| := p$.

Words $\leftrightarrow$ Ferrers shapes in a box.

Example : $\sigma = 0101011110$, so $|\sigma| = 10, |\sigma|_0 = 4, |\sigma|_1 = 6$.



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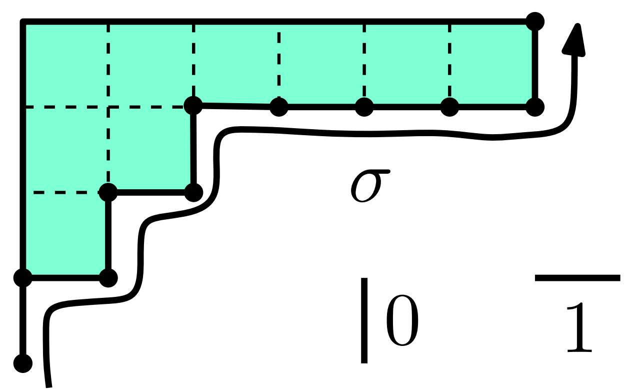
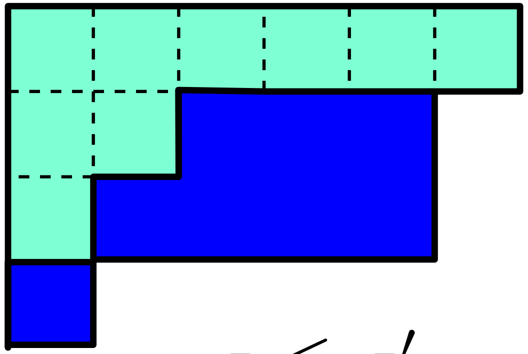


$d(\sigma)$:= the number of boxes in the diagram σ.

σ^* $:=(1-\sigma_p)\cdots(1-\sigma_2)(1-\sigma_1)$

In the example, $d(\sigma) = 9$ and $\sigma^* = 1000010101$.

Words and Shapes



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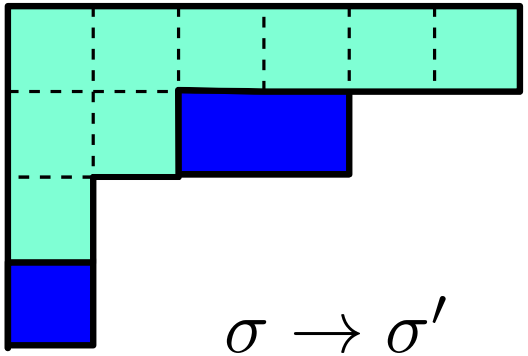

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At most one more box per column

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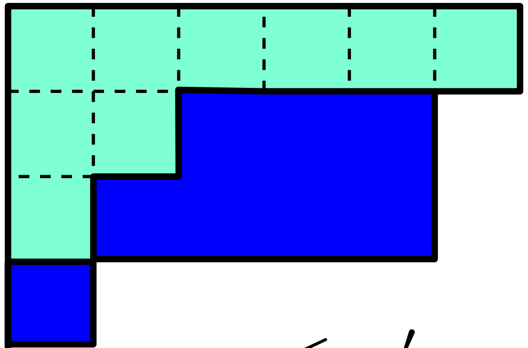

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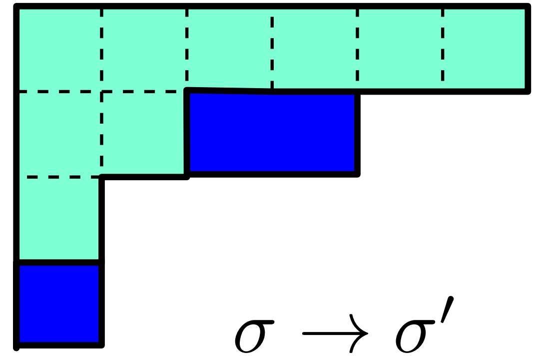

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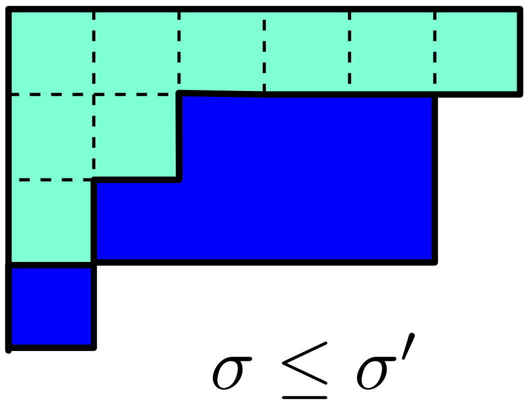
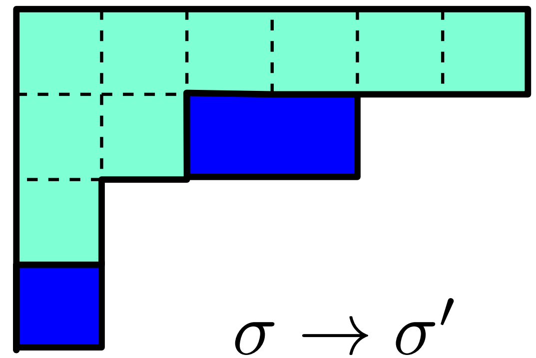
At most one more box per column

A semi standard Young tableau of shape σ and entries bounded by N is a filling of the shape σ by integers in $\{1, \ldots, N\}$ such that entries are strictly increasing in columns and weakly increasing in rows.

Words and Shapes





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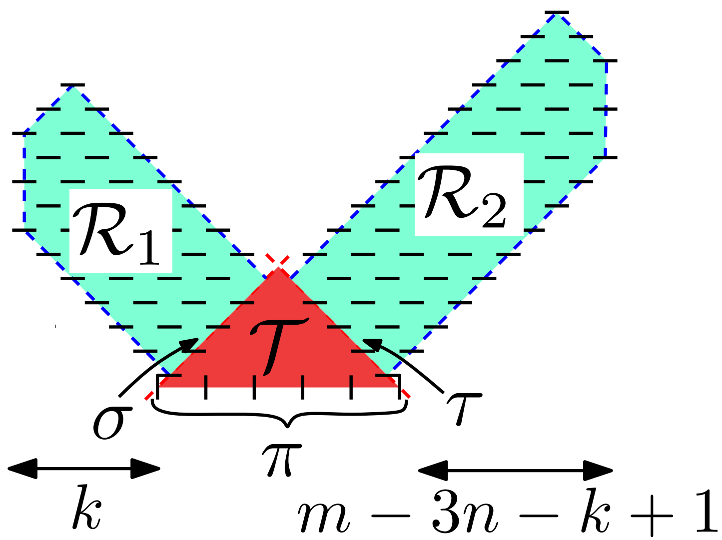
The number of such tableaux is given by $SSYT(\sigma, N)$, an explicit polynomial in N with leading term $\frac{1}{H(\sigma)} N^{d(\sigma)}$.

(Here $H(\sigma)$ is the product of *hook lengths* of the shape σ.)

Regions $\mathcal{R}_1$ and $\mathcal{R}_2$

Proposition [Caselli,Krattenthaler,Lass,N. '05]
Let σ be a word of length $2n$, and $k \in \mathbb{N}$. There is a bijection between *FPLs in* $\mathcal{R}_1(\sigma, k)$ and *semistandard Young tableaux of shape* σ *and length* $n + k$.



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So for $m \geq 3n-1$ (and $k=0$) we obtain :

$$\begin{aligned} A_{\pi \cup m} &= \sum_{\sigma,\tau} |\mathcal{R}_1(\sigma, 0)| \ \cdot \ t^{\pi}_{\sigma,\tau} \ \cdot \ |\mathcal{R}_2(\tau, m-3n+1)| \\ &= \sum_{\sigma,\tau} SSYT(\sigma, n) \cdot t^{\pi}_{\sigma,\tau} \cdot SSYT(\tau^*, m-2n+1) \end{aligned}$$

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Theorem [CKLN '05]

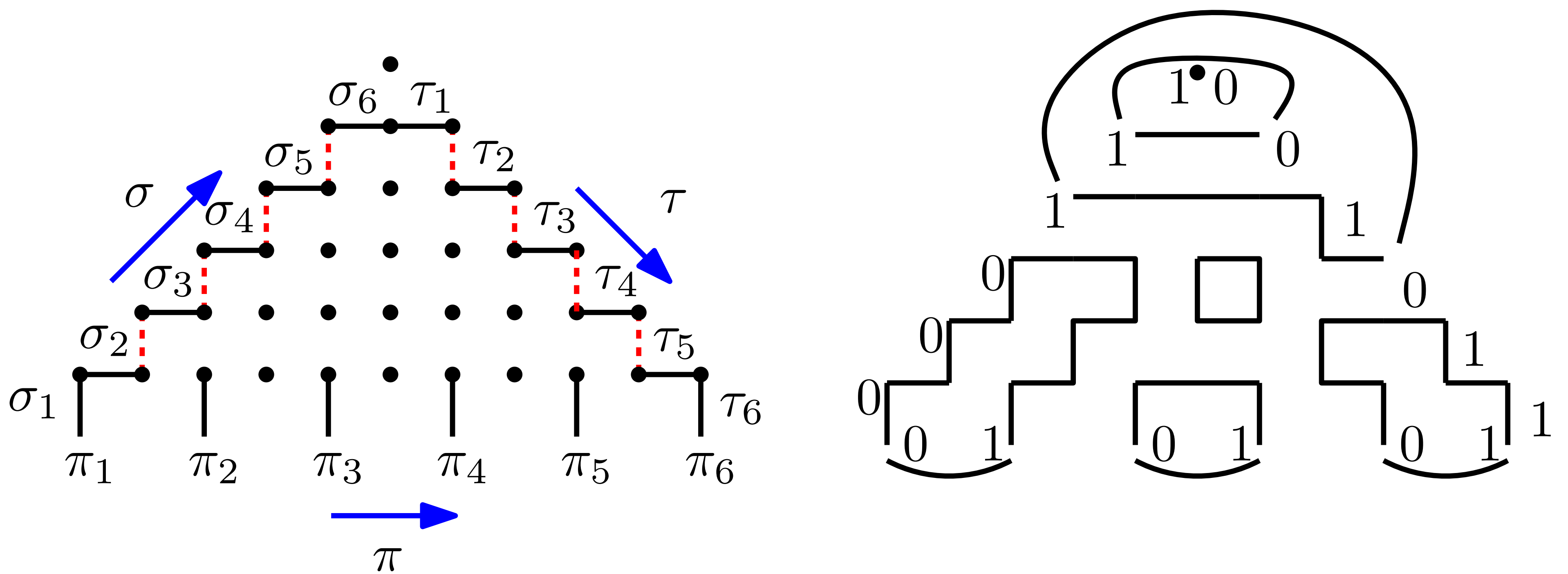
$A_{\pi \cup m}$ is a polynomial function of m for $m \geq 0$

More precisely, $A_{\pi \cup m}$ has leading term $\frac{1}{H(\pi)} m^{d(\pi)}$.

The triangle $\mathcal{T}_n$

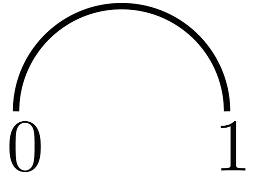
We call TFPLs (Triangular FPLs) the configurations in the triangular counted by $t^{\pi}_{\sigma,\tau}$:

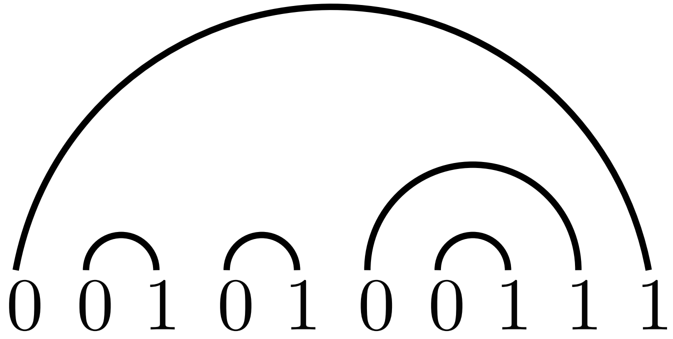
- σ and τ encode the presence of vertical edges on the left and right boundary ;
- π encodes a matching between the lower external edges.

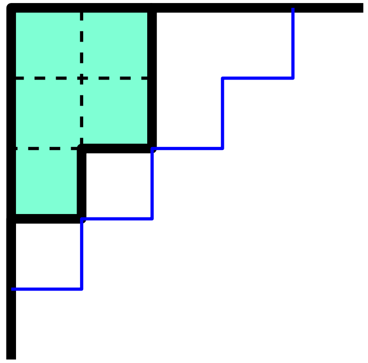


Some more definitions

Given a noncrossing matching π of size n, we can associate to it a word, and thus a Ferrers shape :







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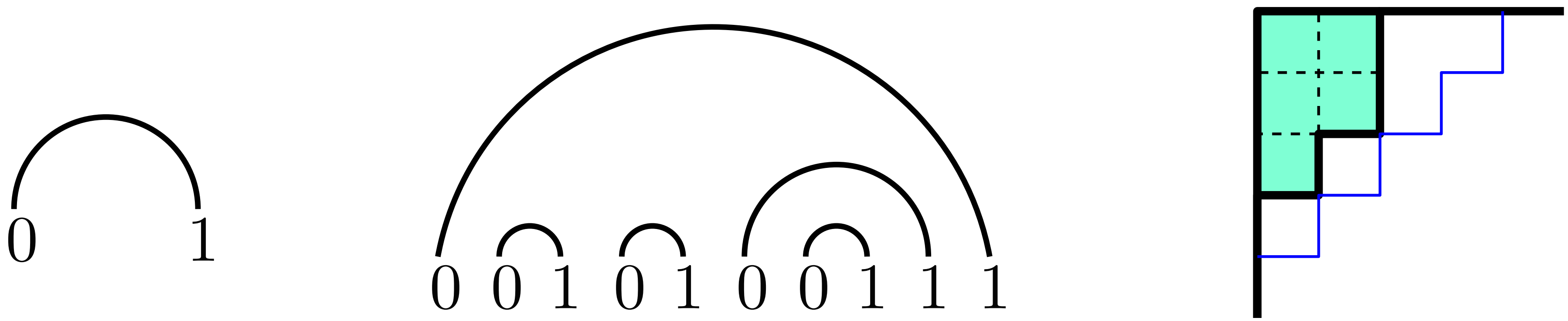

We note $\mathcal{D}_n$ the words w such that $|w|_0 = |w|_1 = n$ and which are smaller than $(01)^n$.

$(\mathcal{D}_n, \leq)$ forms a poset with minimum $\mathbf{0}_n := 0^n 1^n$ and maximum $\mathbf{1}_n := 0101 \cdots 01$.

Some properties of TFPLs

Theorem [N '09]

$$\sum_{\substack{\sigma_1 \in \mathcal{D}_n \\ \sigma \to \sigma_1}} t^{\pi}_{\sigma_1,\tau} = \sum_{\substack{\tau_1 \in \mathcal{D}_n \\ \tau^* \to \tau_1^*}} t^{\pi}_{\sigma,\tau_1}.$$

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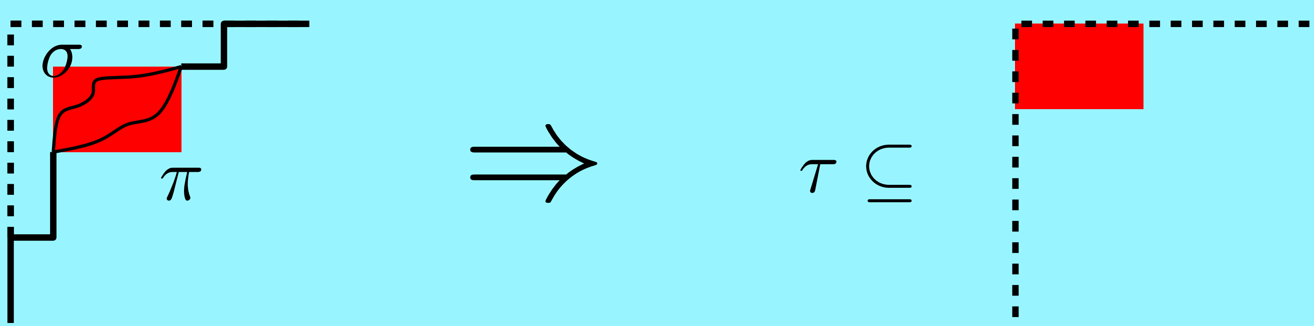

Extremal configurations

Thapper proved another important nonvanishing constraint :

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$$\frac{1}{H(\pi)} = \sum_{\substack{\sigma,\tau \in \mathcal{D}_n \\ d(\sigma)+d(\tau)=d(\pi)}} t^{\pi}_{\sigma,\tau} \cdot \frac{1}{2^{d(\sigma)} H(\sigma)} \cdot \frac{1}{2^{d(\tau)} H(\tau)}$$

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We name extremal the TFPL with boundaries $\{\sigma, \pi, \tau\}$ verifying $d(\sigma) + d(\tau) = d(\pi)$.

Littlewood–Richardson coefficients

Let λ, μ, ν be partitions, and $\Lambda(x)$ be the ring of symmetric functions of the variables $x_1, x_2, \ldots$. The Schur functions $s_\lambda(x)$ can be defined as

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Schur functions form a basis of $\Lambda(x)$. We can expand $s_\mu(x)s_\nu(x)$ on this basis, where the coefficients $c^\lambda_{\mu,\nu}$ are often called the Littlewood-Richardson (LR) coefficients.

$$s_\mu(x)s_\nu(x) = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda(x)$$

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We have

$$c^{\lambda}_{\mu,\nu} = 0 \text{ unless } d(\lambda) = d(\mu) + d(\nu) \text{ and } \mu, \nu \subseteq \lambda$$

We have also, if $s_\lambda(x, y)$ is the symmetric function s_λ in the variables $x_1, x_2, \ldots, y_1, y_2, \ldots$

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From this one can obtain the identity

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As a consequence, there exist $a_{\sigma\tau} > 0$ such that, for any $\pi \in \mathcal{D}_n$,

$$\sum_{\sigma,\tau} a_{\sigma\tau} c^{\pi}_{\sigma,\tau} = \sum_{\sigma,\tau} a_{\sigma\tau} t^{\pi}_{\sigma,\tau} \qquad (E)$$

in which σ, τ run through words such that $d(\sigma) + d(\tau) = d(\pi)$.

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Thanks to equation (E), we need only prove that

$$c^{\pi}_{\sigma,\tau} \leq t^{\pi}_{\sigma,\tau}$$

for such extremal σ, τ, π.

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We use here Knutson-Tao puzzles.

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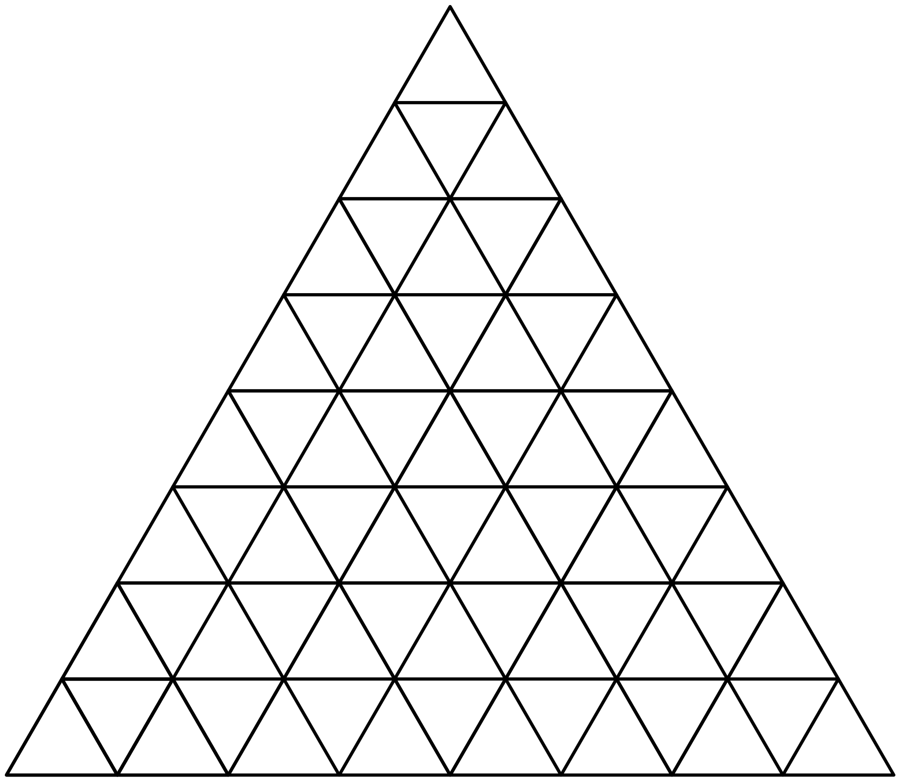


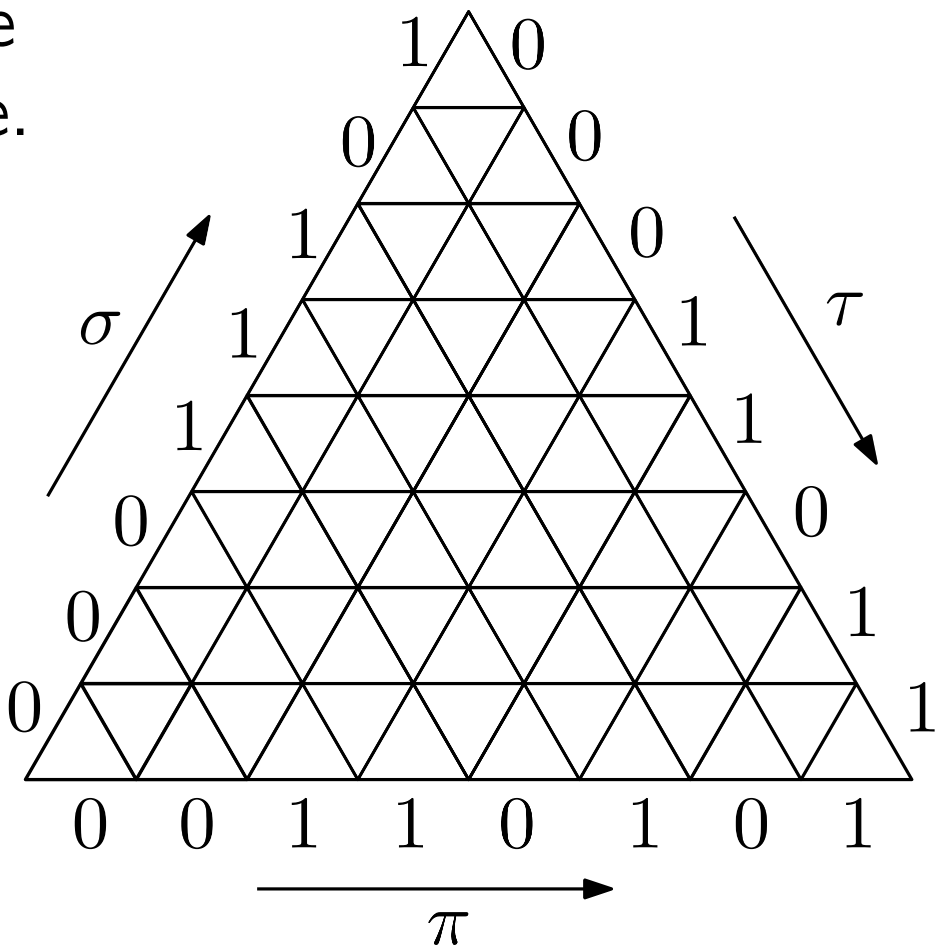
Littlewood–Richardson coefficients

There are many objects that are counted by LR-coefficients. We use here Knutson-Tao puzzles.

Consider a triangle of size $2n$ on the triangular lattice.

Fix $\sigma, \pi, \tau \in \mathcal{D}_n$, and label the boundary edges of the triangle.

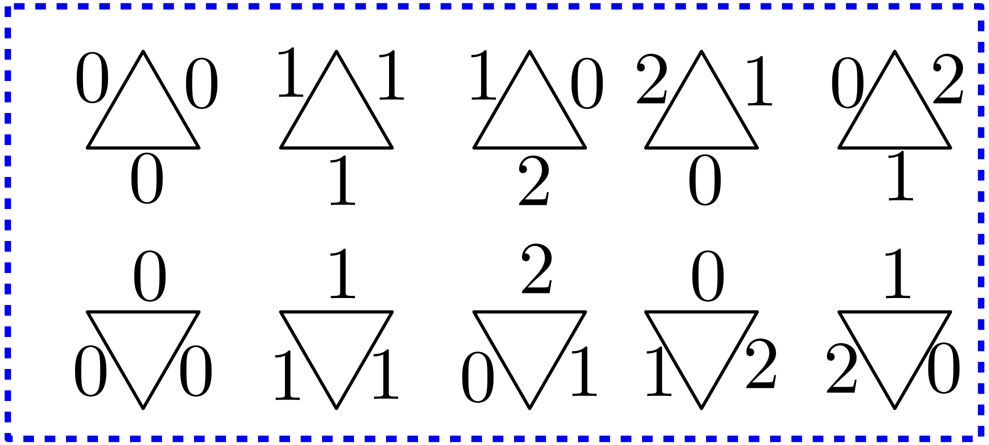
$$\sigma = 00011011$$
$$\tau = 00011011$$
$$\pi = 00110101$$



Knutson–Tao puzzles

A Knutson-Tao puzzle with boundary data σ, π, τ is a labeling of each edge of the triangle by $0, 1$ or 2, such that :

- the labels on the boundary are given by σ, π, τ ;
- on each unit triangle, the induced labeling must be among :



"Only 0s, only 1s, or $0, 1, 2$ counterclockwise"

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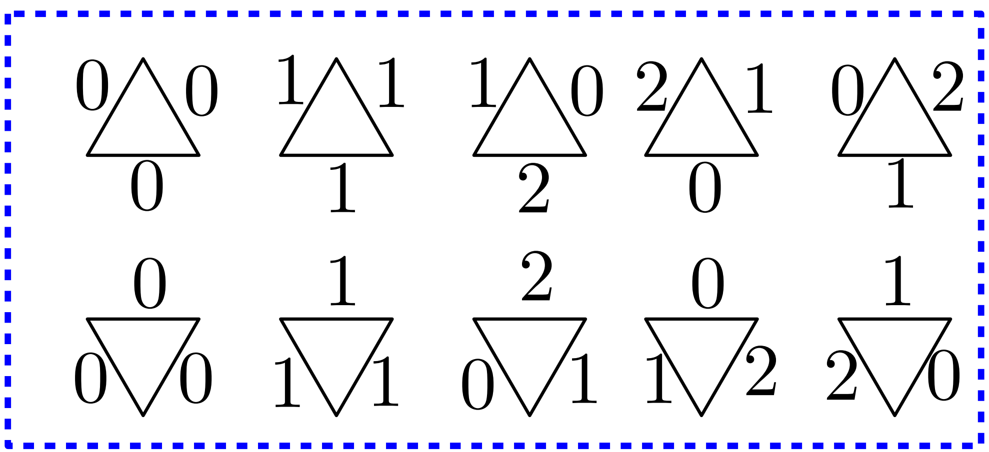

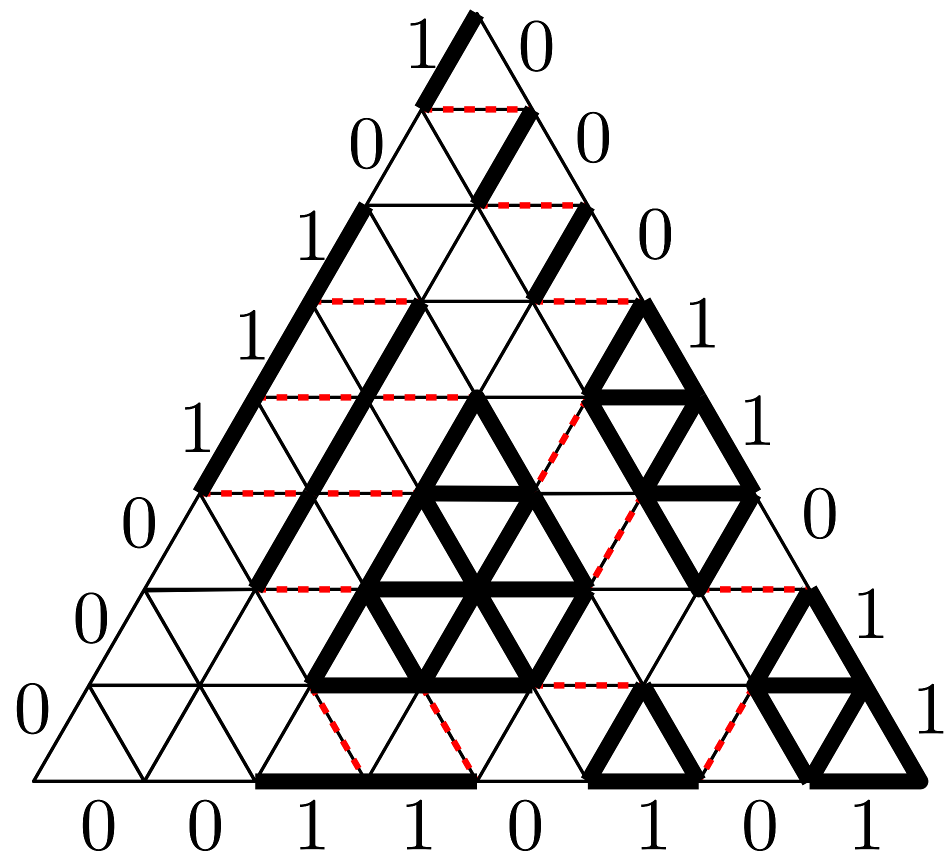

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Theorem [Knutson, Tao '03][K., T. and Woodward '03]
Let $\sigma, \tau, \pi \in \mathcal{D}_n$. Then the number of KT-puzzles with boundary data σ, π, τ is equal to the LR coefficient $c^{\pi}_{\sigma,\tau}$.

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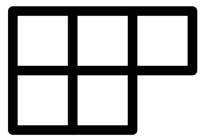
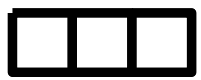
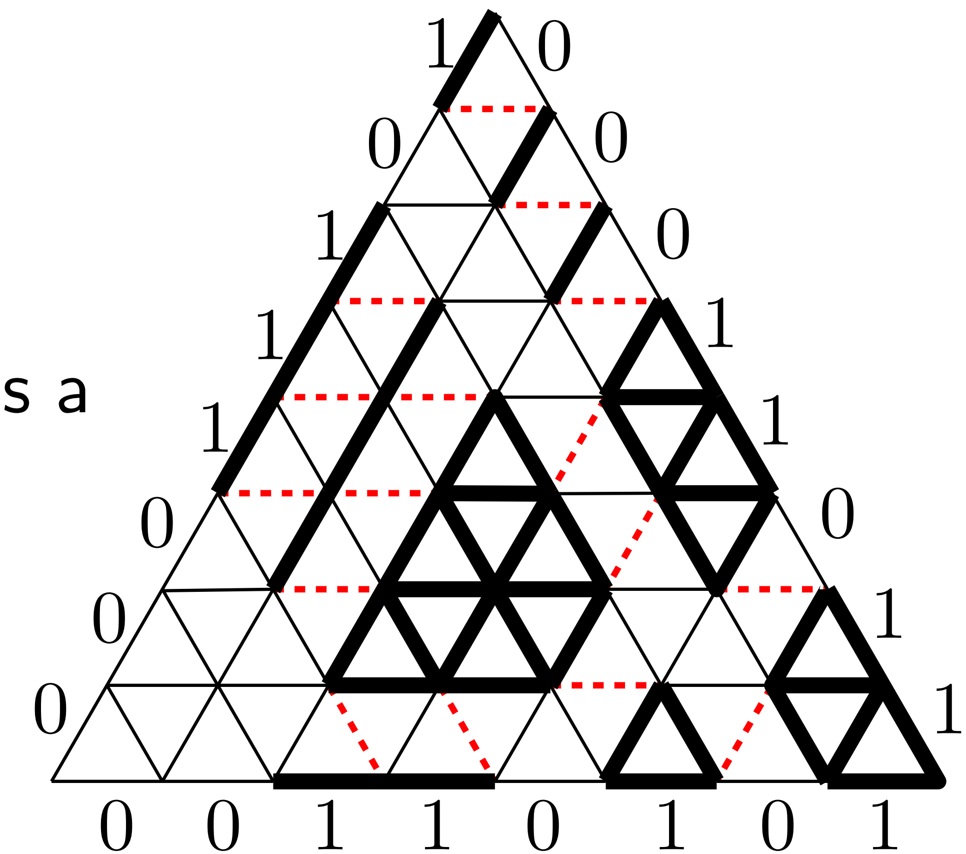
$\lambda =$ 

$\mu =$ 

$\nu =$ 

Then $c^{\lambda}_{\mu,\nu} = 1$ because there is a unique puzzle in this case.



From KT puzzles to TFPL configurations

We fix $\sigma, \pi, \tau \in \mathcal{D}_n$, such that $d(\sigma) + d(\tau) = d(\pi)$. We will define a map Φ.

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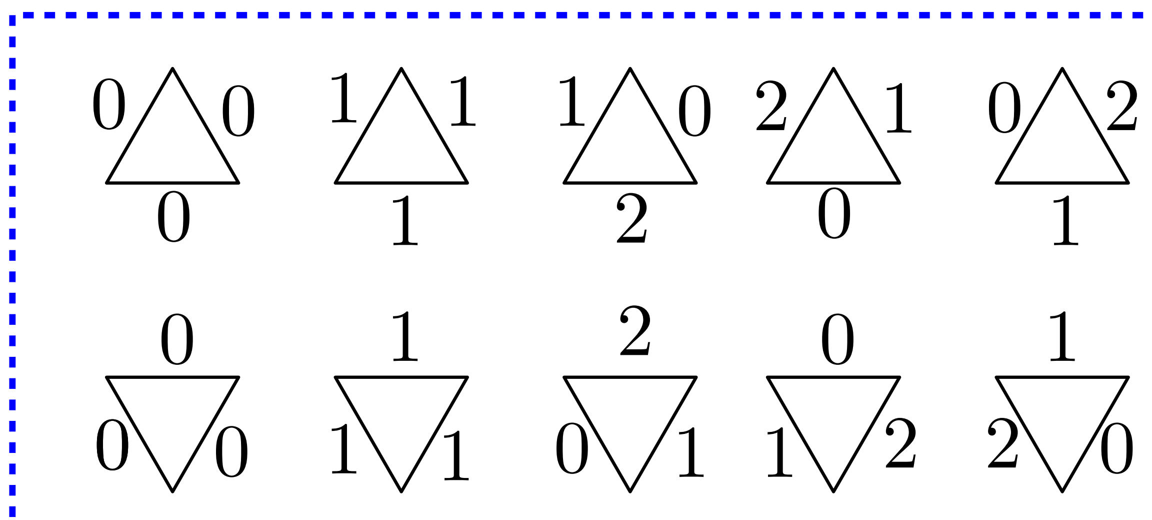
The map is local : it changes every small labeled triangle of a puzzle to a portion of a path in a TFPL configuration.



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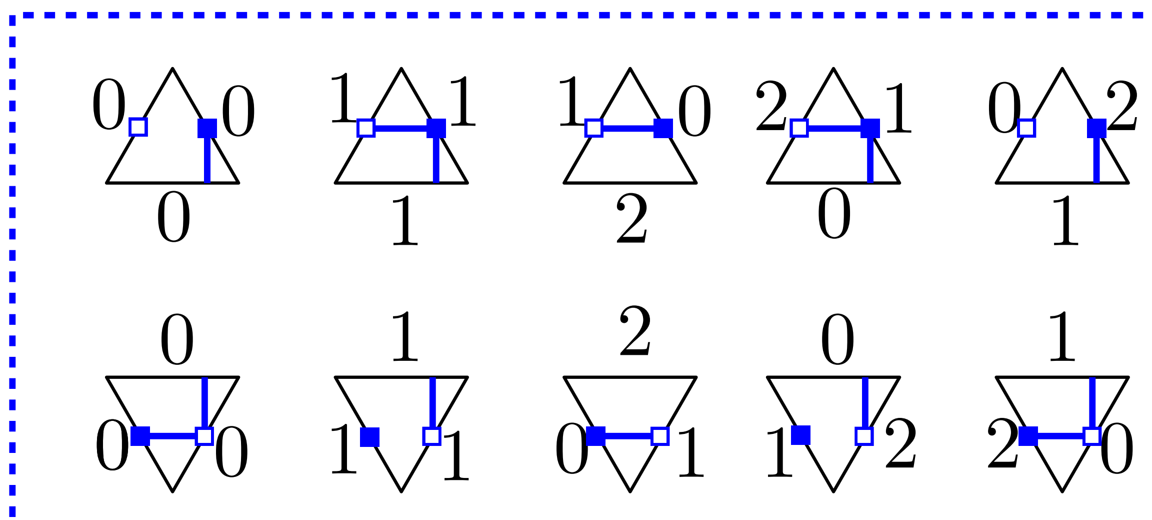
Definition of Φ



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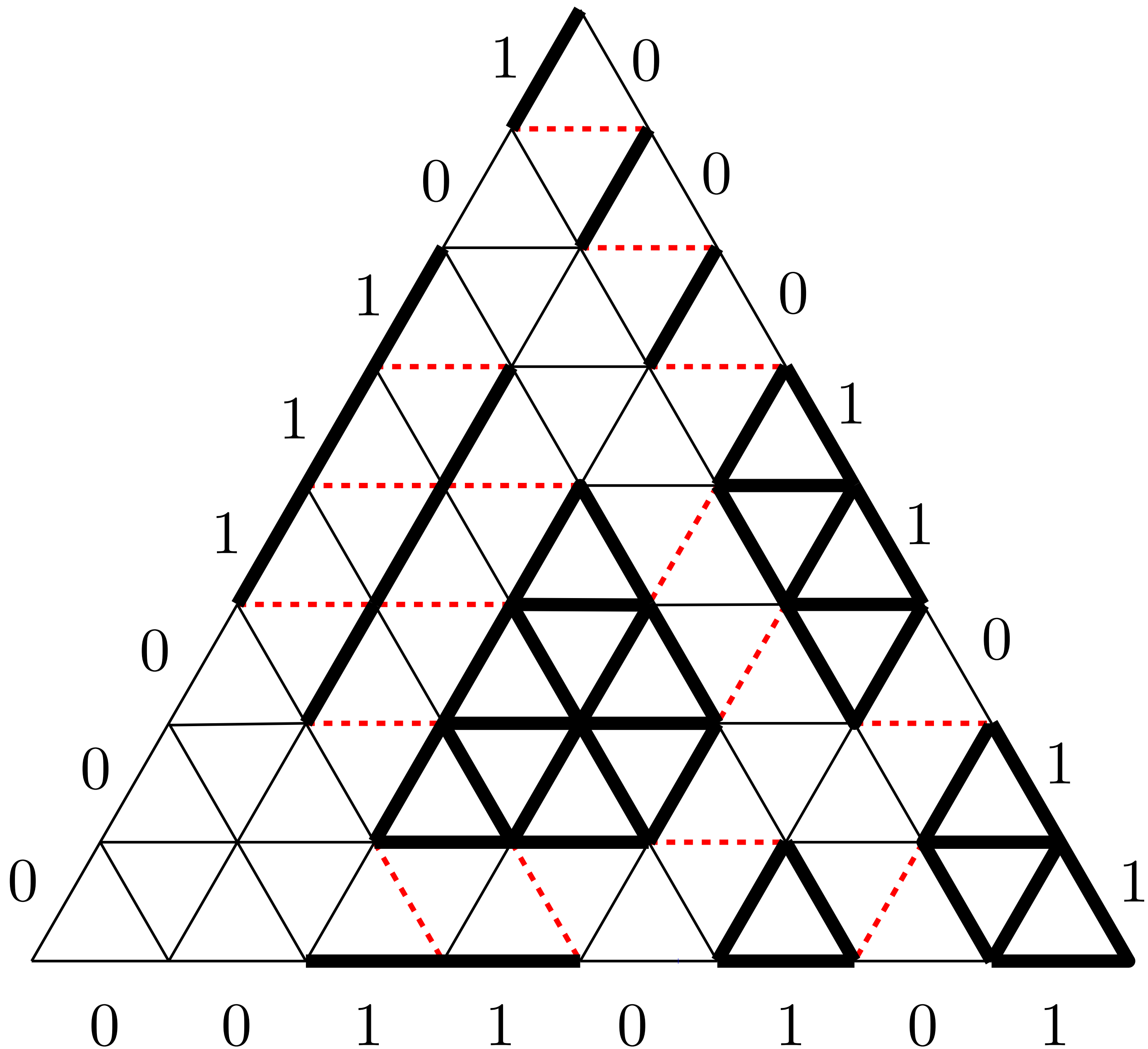

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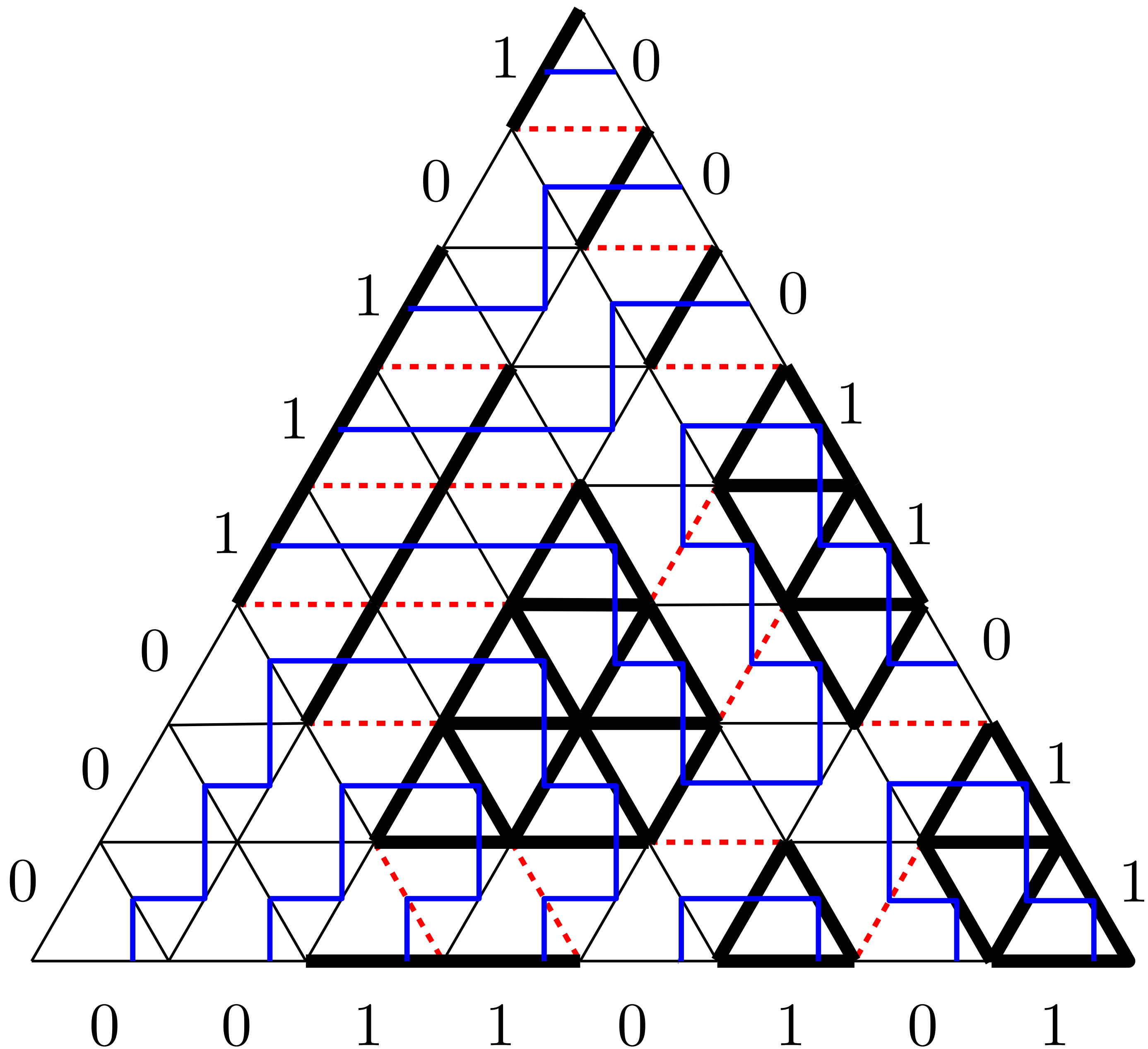

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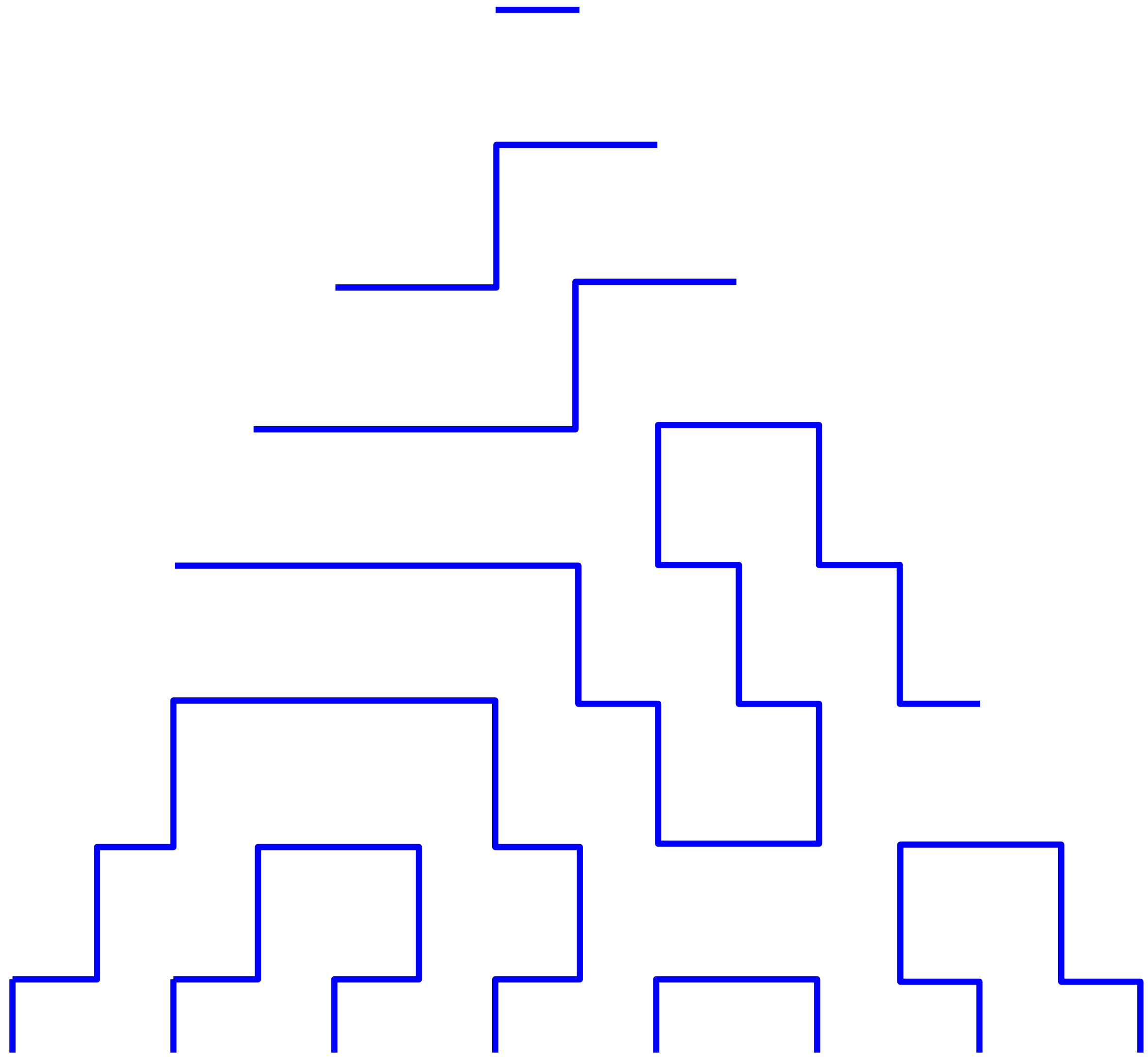

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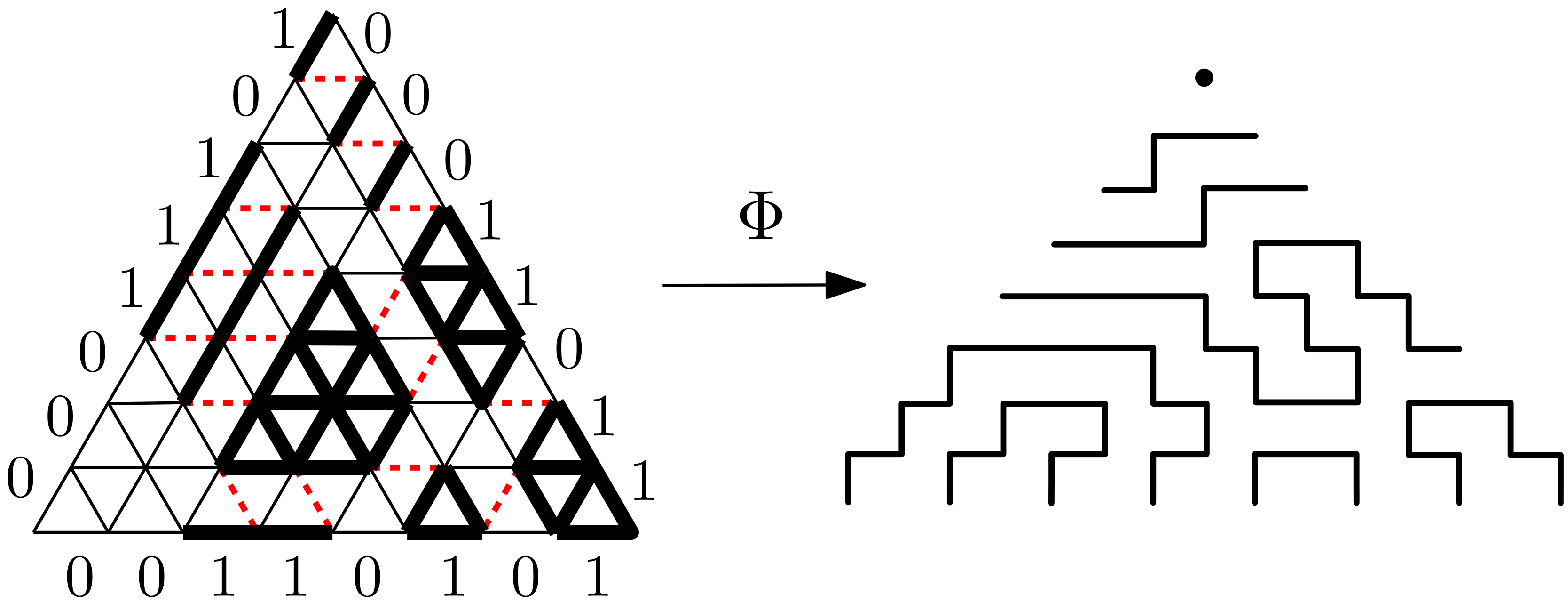

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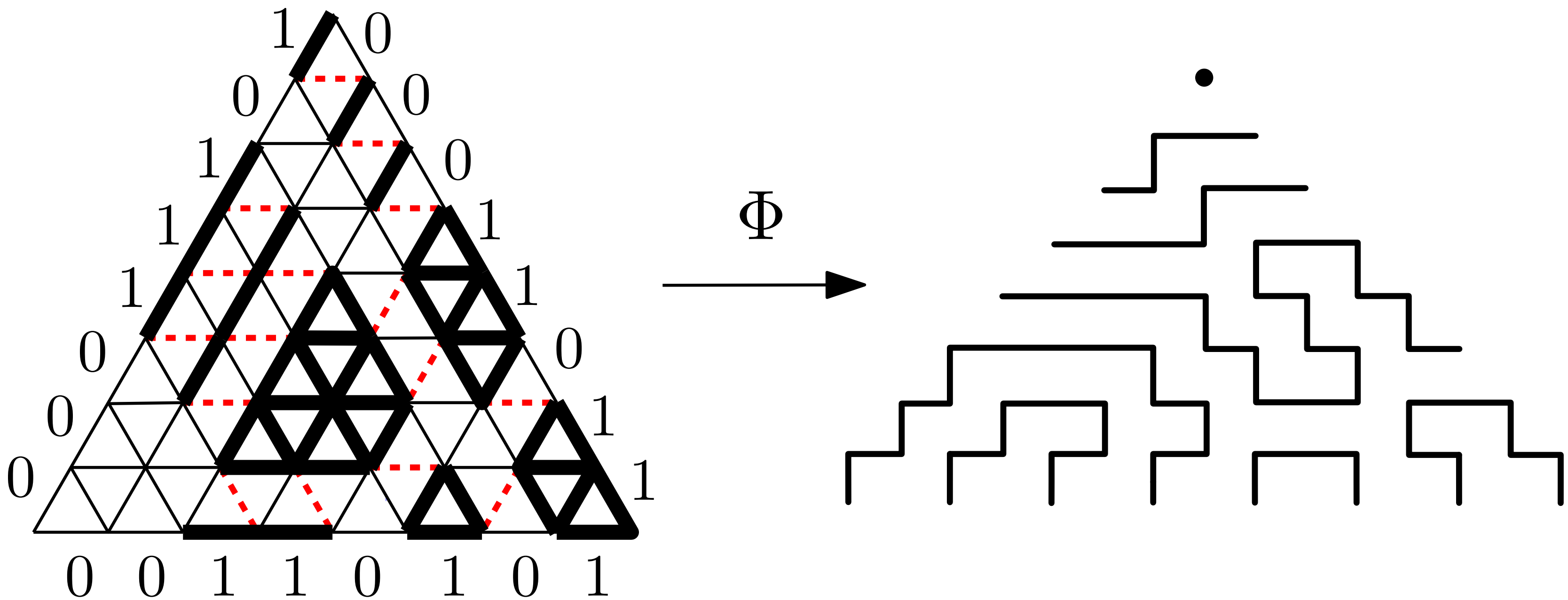

If P is a KT-puzzle with boundaries σ, τ, π one must show :

1. Φ is well defined :
   (a) the vertices of $\Phi(P)$ are of degree 2 ,
   (b) $\Phi(P)$ verifies the boundary conditions σ, τ.
   (c) the connectivity of external edges given by π is respected.
2. Φ is injective.

Further questions

1) To compute A_X, one needs all $t^{\pi}_{\sigma,\tau}$ beyond the case $d(\pi) = d(\sigma) + d(\tau)$. So the question is :
What is the general value/interpretation of $t^{\pi}_{\sigma,\tau}$?
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3) Related work (with T. Fonseca) : results + conjectures about the polynomials $A_{\pi\cup m}$, pointing to a combinatorial reciprocity phenomenon.