On k-crossings and k-nestings of permutations

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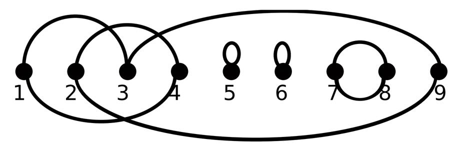
Permutations as arc annotated sequences

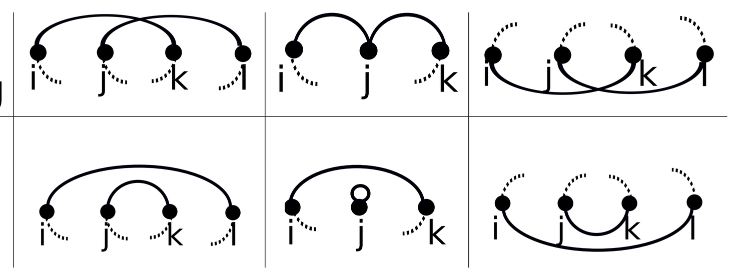
We can draw a permutation $\sigma \in \mathfrak{S}_n$ by visualizing its disjoint cycles. We follow Corteel [2] and draw ascents on top, and descents on the bottom. This is a permutation as an **arc annotated sequence**.

Example We draw the permutation $\sigma = (1\,3\,9\,2\,4)(5)(6)(7\,8)$:



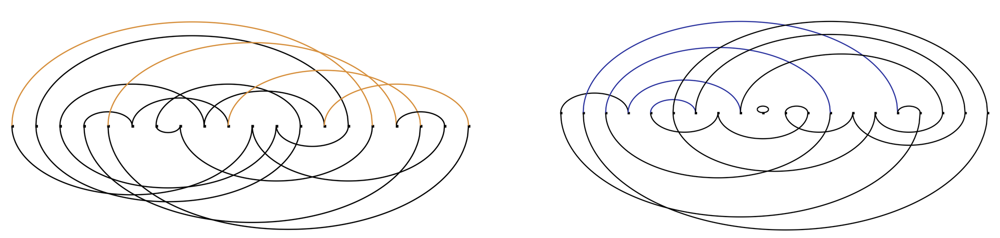
Crossings and nestings

In an arc annotated sequence natural **nesting** and **crossing** structures arise. This is what we consider to be nestings and crossings in a permutation.

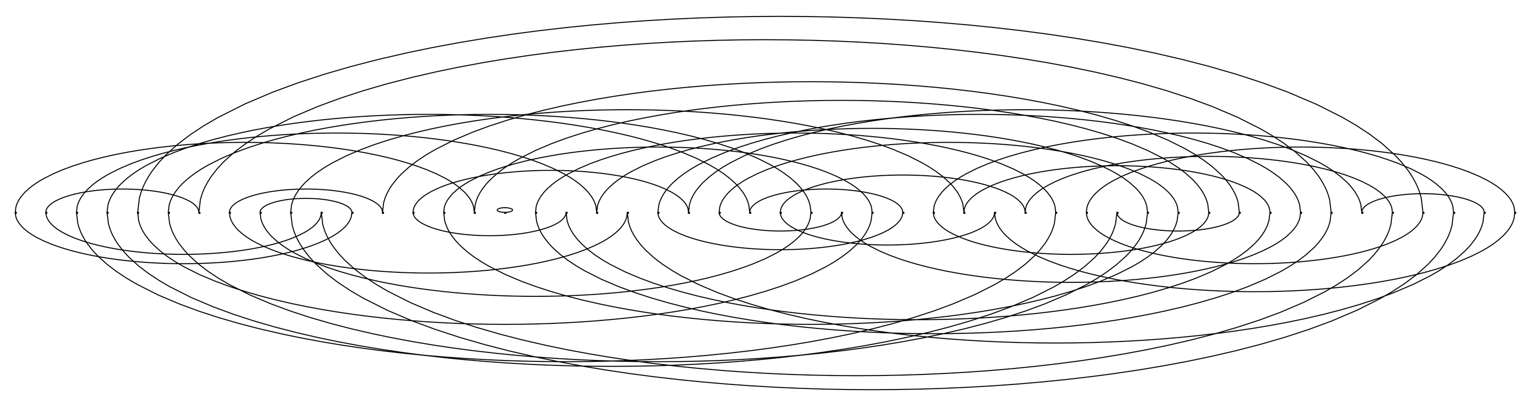


Definition We define a **k-crossing (k-nesting)** in a permutation as a set of k mutually crossing (nesting) arcs in the arc annotated sequence. The **crossing number** of a permutation is the largest k for which the permutation contains a k-crossing. The nesting number is similarly defined.

Example Consider the following permutations of $\{1, 2, \ldots, 20\}$ with a 4-crossing highlighted in gold, and a 4-nesting highlighted in blue.



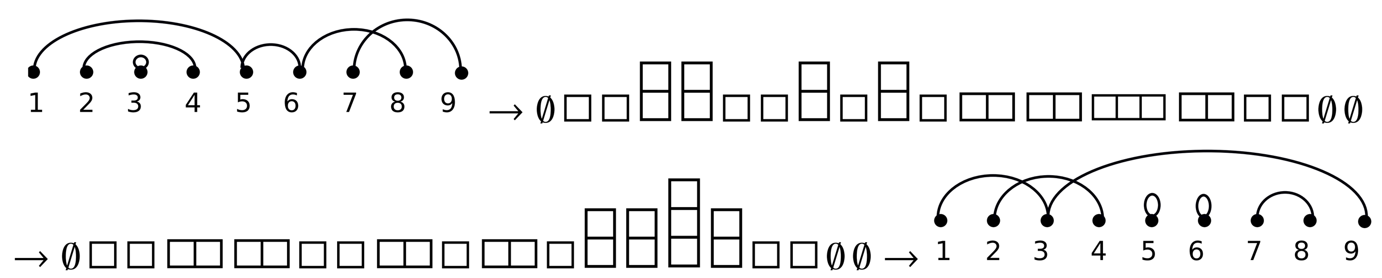
The following permutation has a nesting number of 6 and a crossing number of 6. Can you find a 6-crossing or a 6-nesting?



Context

Chen, Dend, Du, Stanley and Yan [1] studied k-crossings and k-nestings in **matchings** and **set partitions**. They showed, via a bijection ϕ_1 with vacillating (hesitating) tableaux, that there was a symmetric joint distribution for the crossing and nesting numbers in both matchings and partitions.

Example Here is an example of their bijection on the set partition $1568 - 24 - 3 - 79$.



Corteel [2] showed **crossings** and **nestings** appear in equal numbers in permutations, but she does not consider k-crossings and k-nestings. We extend Chen *et al.*'s result to permutations using Corteel's definitions.

Equidistribution

Definition The *degree sequence*, D_σ, of a permutation σ is the sequence of indegree and outdegrees of the vertices on the top half ($\sigma+$) of the permutation:

$$D_\sigma \equiv (indegree_{\sigma+}(i), outdegree_{\sigma+}(i))_{i=1}^{n}$$

We also define $\overline{D_\sigma}(i) \equiv (indegree_{\sigma-}(i), outdegree_{\sigma-}(i))_{i=1}^{n}$ as the *lower* degree sequence.

There are only 4 types of vertices possible in each permutation, and as such each degree sequence is made up of (indegree, outdegree) pairs from the set $\{(1,0),(0,1),(0,0),(1,1)\}$:

Type	vertex i	$D_\sigma(i)$	$\overline{D_\sigma}(i)$	Type	vertex i	$D_\sigma(i)$	$\overline{D_\sigma}(i)$
opener		(0,1)	(1,0)	upper transient		(1,1)	(0,0)
closer		(1,0)	(0,1)	lower transient		(0,0)	(1,1)

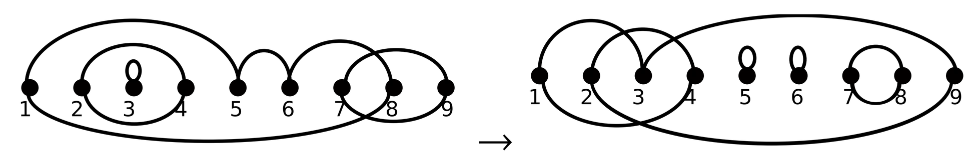
We give our main result:

Theorem Let $NC(n, i, j, D)$ be the number of permutations of n with crossing number i and nesting number j, and degree sequence specified by D. Then,

$$NC(n, i, j, D) = NC(n, j, i, D).$$

Proof To prove this, we adapt the Chen *et al.* bijection for set partitions to make an involution $\psi : \mathfrak{S}_n \to \mathfrak{S}_n$ that preserves the degree sequence and swaps nesting and crossing numbers. Essentially, we treat the top and bottom independently. Their bijection preserves openers and closers, hence the resulting pair of partitions forms a consistent permutation arc annotated sequence.

Example



Enumeration

Definition Let $C(n, k)$ be the number of permutations of length n with crossing number k.

The following table gives $C(n, k)$ for small values of n and k.

$n \backslash k$	1	2	3	4	5	6
1	1					
2	2					
3	5	1				
4	14	10				
5	42	76	2			
6	132	543	45			
7	429	3904	701	6		
8	1430	29034	9623	233		
9	4862	225753	126327	5914	24	
10	16796	1839540	1644215	126834	1415	
11	58786	15679886	21604496	2521165	52347	120

Notice, the sequence $C(n, 1)$ is the Catalan numbers, and is known to count noncrossing permutations of other types. We can also determine the number of permutations with a maximum crossing. Across all permutations of size n the maximum crossing possible is of size $\lceil n/2 \rceil$.

Theorem

$$C\left(n, \left\lceil \frac{n}{2} \right\rceil\right) = \begin{cases} m! & \text{if } n = 2m+1; \\ (m-1)!(2m^2-1) + 2(m!) - 1 & \text{if } n = 2n. \end{cases}$$

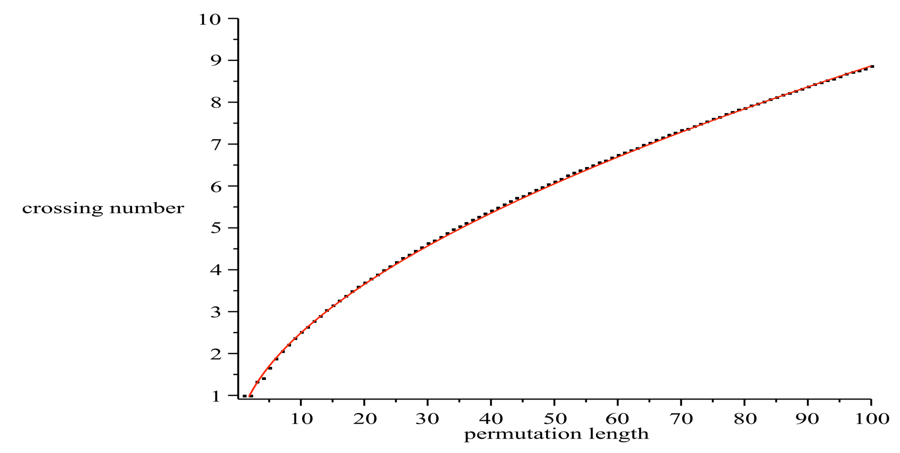
A very small proportion of permutations are either maximum crossing ($\leq 2^{\frac{n+1}{2}}!/n!$) or noncrossing ($\approx 4^n n^{3/2}/n!$)

Open questions

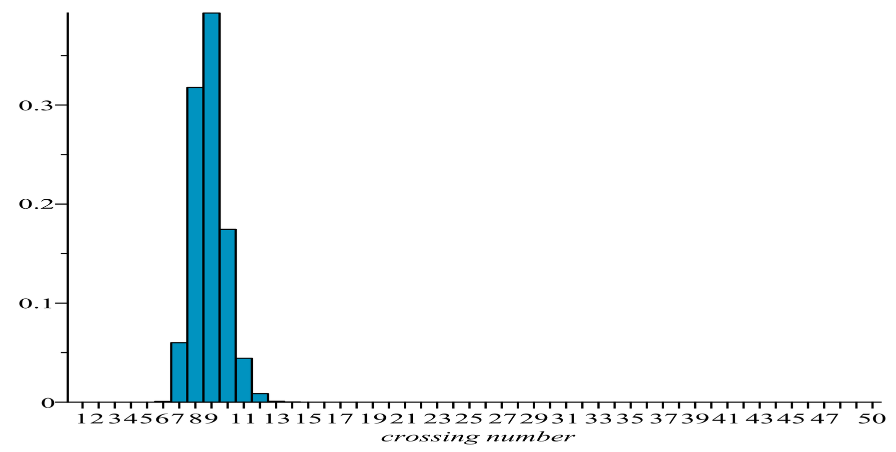
- **What is the average crossing number?**

Conjecture The average crossing number for a permutation of length n is $\sim \frac{7}{10} n^{\frac{\sqrt{3}}{\pi}}$.

Evidence:



- **What is the distribution of the crossing number?** Chen *et al.* [1] show that nestings in fixed point free involutions create decreasing sequences, and hence the Tracy-Widom distribution arises. By taking a sample of size 1000 of permutations of length 100, we see a similar shape, and that the most common crossing number is quite small (~ 9).



- **Can we find an involution avoiding tableaux and fillings of Ferrers diagrams to illustrate the symmetric joint distribution of crossing and nesting statistics?**
- **For fixed k, is generating function for k-noncrossing permutations D-finite?**
- **Which sub-classes of permutations preserve equidistribution of the nesting and crossing numbers?** Any class that preserves the degree sequence will preserve the equidistribution, for instance involutions, and fixed point free involutions. What are others?
- **How does this relate to other statistics on permutations?**

References and acknowledgments

[1] W. Y. C. Chen, E. Y. P. Dend, R. R. X. Du, R. P. Stanley, and C. H. Yan, *Crossings and nestings of matchings and partitions*, Trans. Amer. Math. Soc. **359**, 1555-1575, 2007.

[2] S. Corteel, *Crossings and alignments of permutations*, Adv. in App. Math. **38**, 149-163. 2007.

[3] A. de Mier, *k-noncrossing and k-nonnesting graphs and fillings of Ferrers diagrams*, Combinatorica **27**, no. 6, 699–720, 2007.

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