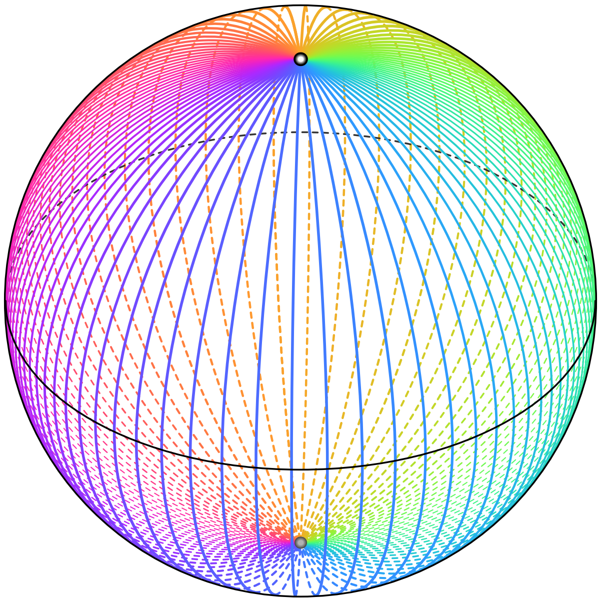
Posets and Curvature



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A caution and a request

Some of my recent work is fairly combinatorial:

I used möbius inversion and factorizations in incidence algebras to push curvature around in high dimenisional cell complexes (§11 in "Constructing nonpositively curved spaces and groups" LMS LNS)

I established multivariable polynomial identities by summing over the poset of hypertrees and used formal power series and exponential generating functions to do cohomology calculations for the group of loops. (a series of papers with J.Meier et al.)

Caution: this talk is not as combinatorial. In particular I ask noncombinatorial questions of combinatorial objects.
Request: indulgence.

Main goal and a challenge problem

My main goal is to explain how I view combinatorial objects through a metric lens. Let's start with a challenge problem. Solutions at the end of the talk.

Let Γ be an n-vertex planar graph in which all but one vertex has degree 5 and every bounded region is a triangle. Let k be the degree of the exceptional vertex and let the unbounded region be an ℓ-gon.

1. What are the possible values for n?
2. What are the possible values for k?
3. What are the possible values for ℓ?
4. What is the relationship between n, k, and ℓ?
5. Classify *all* such graphs.

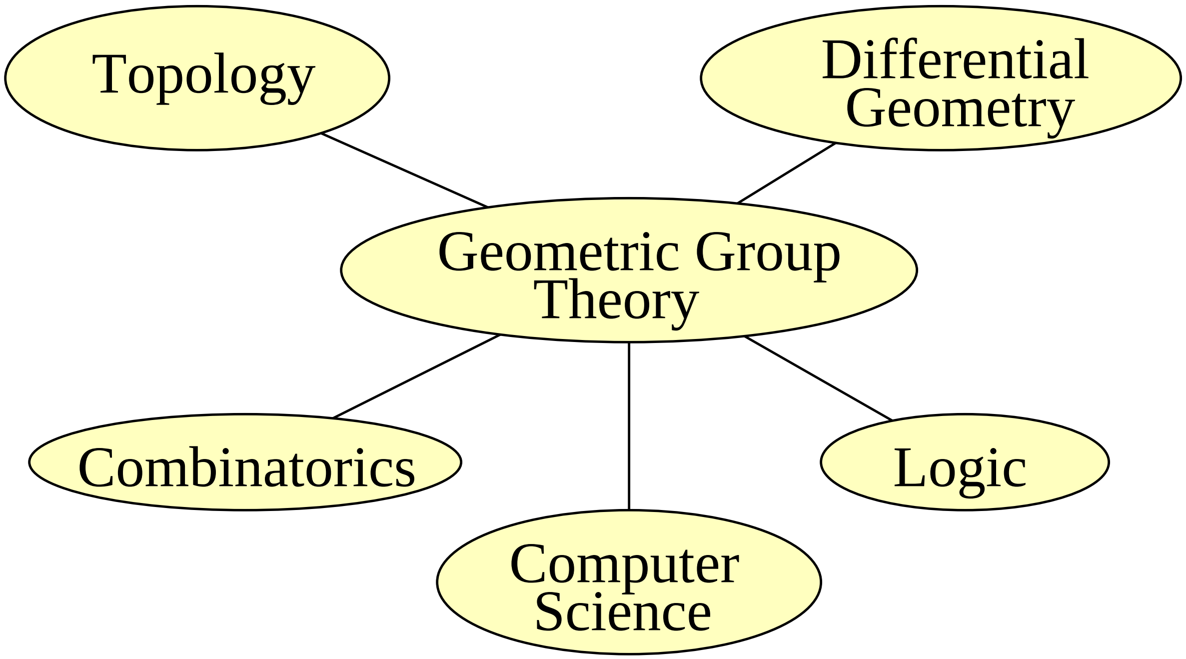
Outline

I. Geometric group theory
II. Posets and groups
III. Orthoschemes and complexes

Ten years ago Tom Brady showed me a new classifying space for the braid groups and I suggested a nice piecewise Euclidean metric. We conjectured that the result has good curvature properties and this launched a series of papers with a computational focus.

After a quick primer on geometric group theory, I'll explain the space, the metric and the associated posets. Along the way I'll suggest a metric to use on order complexes and highlight a combinatorially defined poset substructure that is interesting from a metric viewpoint.

I. Geometric group theory



Geometric group theory is built around two key ideas:

- Finitely generated groups have intrinsic metrics.
- Groups with “non-positively curved” metrics are well-behaved.

The Big Bang: Gromov hyperbolic groups

A metric space is δ-*hyperbolic* when every geodesic triangle is δ-thin (as in the hyperbolic plane).

A finitely generated group has a metric determined by calculating distances in its Cayley graph. Gromov highlighted that although the metric changes when the generating set changes, they are roughly equivalent (quasi-isometric).

A f.g. group is (Gromov) *hyperbolic* when its intrinsic metric in δ-hyperbolic for some $\delta \geq 0$.

Thm(Gromov 85): A f.g. group is hyperbolic iff its word problem can be solved in linear time.

Good, True and Beautiful

In the same way that Socrates hints that the good, the true and the beautiful are classes that (roughly) coincide, geometric group theorists tend to (roughly) believe the following:

Metatheorem: Good geometry and topology = good algebra and combinatorics = good algorithms and computational properties.

For example, the groups with well-behaved intrinsic geometry are those in which efficient algorithms are possible and vice versa. Despite its flaws, this has been an enormously fruitful initial intuition.

PE complexes and PS complexes

Def: Roughly speaking a *piecewise Euclidean* complex K is a quotient of a disjoint union of Euclidean polytopes by isometric face identification and a *piecewise spherical* complex is similarly defined using spherical polytopes.

Def: The *link* of a vertex in a Euclidean polytope is the collection of unit vectors that point into the polytope. The link of a vertex in a PE complex is the natural PS complex obtained by gluing together the vertex links in each of the individual polytopes. Faces also have links defined using unit vectors orthogonal to their affine hulls.

Rem: Every PS complex is the link of a vertex in a PE complex.

Non-positive curvature

Non-positive curvature is usually defined using comparison triangles but in a PE complex the definition can be rewritten in terms of short geodesic loops.

Def: A *geodesic path* is an isometric embedding of an metric interval. A *geodesic loop* is an isometric embedding of a metric circle. A path / loop is *locally geodesic* if every point belongs to an open subinterval which is geodesic. A local geodesic loop is *short* when it has length less than 2π.

Ex: [Equitorial examples in $\mathbb{S}^2$]

Def/Thm: A PE complex is *non-positively curved* when the link of every face is a PS complex with no short local geodesic loops.

Cube complexes

Let K be a PE complex built out of cubes. Gromov provided a very simple test to determine whether or not it is NPC.

Gromov's Link condition: A cube complex is NPC iff the link of each vertex is a flag simplicial complex.

Recall that a simplicial complex is *flag* if every complete subgraph is the 1-skeleton of a simplex (= every non-simplex contains a non-edge).

Because this is easy to test, cube complexes have been used extensively in geometric group theory. Testing complexes built out of other shapes is hard but sometimes doable [Elder-M].

Charney-Davis conjecture

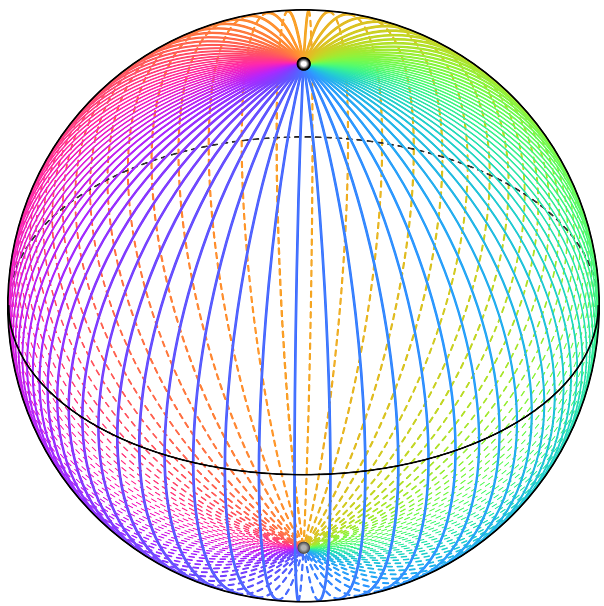
The Charney-Davis conjecture states that for every flag triangulation of an odd dimensional sphere, a linear inequality involving the face numbers should hold. Its origin is a conjecture about euler characteristics of NPC manifolds.

Surfaces are nonpositively curved iff their euler characteristic is nonpositive and $\chi(X \times Y) = \chi(X) \times \chi(Y)$, so NPC manifolds in dimension $2n$ should have $\chi(M)(-1)^n \geq 0$ (Hopf,Chern).

Charney and Davis noted that NPC cubings of manifolds have flag triangulations of spheres as their vertex links and the euler characteristics of NPC cube complexes can be found by summing a curvature contribution from each vertex written as a linear combination of the face numbers of its link.

II. Posets and groups

Def: In any metric space we say that z is *between* x and y when $d(x,z)+d(z,y)=d(x,y)$. The collection of all points between x and y is the *interval* $[x,y]$. Intervals are bounded partially ordered sets with an ordering $z \leq w$ iff $d(x,z)+d(z,w)+d(w,y)=d(x,y)$.



Groups $\Rightarrow$ Intervals

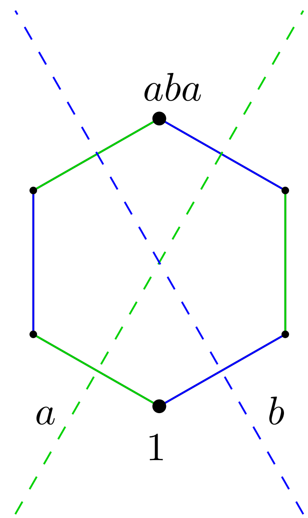
The natural measure of length in a directed graph Γ is the length of the shortest directed combinatorial path from x to y. We use this natural (partial asymmetric) metric to define intervals in Cayley graphs using the same equations.

Since posets can be recovered from their Hasse diagrams, let $[g, h]$ denote the edge-labeled directed graph inside the Cayley graph that is the union of shortest directed paths from v_g to v_h. It is also the Hasse diagram of the poset order on $[g, h]$.

Rem: Because Cayley graphs are homogeneous, the interval $[g, h]$ is isomorphic (as an edge-labeled directed graph) to the interval $[1, g^{-1}h]$. Thus it is sufficient to consider intervals of the form $[1, g]$.

Example 1

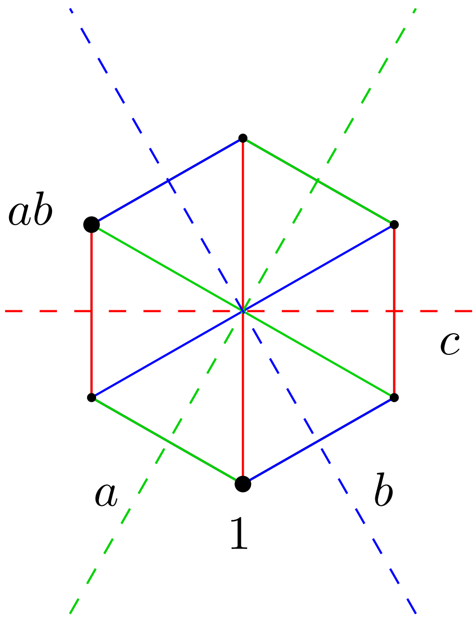
If G is the symmetric group, S is the set of adjacent transpositions and g is the "half-flip" permutation, then the interval $[1, g]$ is the 1-skeleton of the permutahedron with a Morse function.

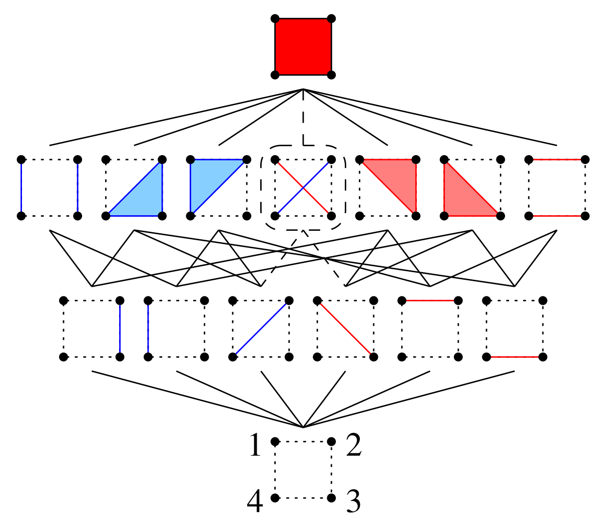


Question: What other bounded graded posets arise as intervals inside "nice" (finite) groups with reasonable generating sets?

Example 2

If G is the symmetric group, S is the set of all transpositions, and g is an n-cycle, then the interval $[1, g]$ is the non-crossing partition lattice NC_n.





Intervals $\Rightarrow$ Groups

Let $[1, g]$ be an interval in the Cayley graph of an S-generated group G. We construct a new group G_g as the largest group generated by S containing $[1, g]$ as part of its own Cayley graph. In other words, G_g is defined by only adding those relations that are visible inside the interval $[1, g]$.

Def: Let $\mathcal{R}$ be the words corresponding to the directed geodesic paths from v_1 to v_g in the $Cay(G, S)$, i.e. $\mathcal{R}$ is the set of *minimal length positive factorizations* of g over S. It is an easy exercise to show that $\langle S \mid u = v, \text{ for all } u, v \in \mathcal{R} \rangle$ is a presentation for G_g. We say G_g is an *interval group* obtained by *pulling G apart at g*.

Examples

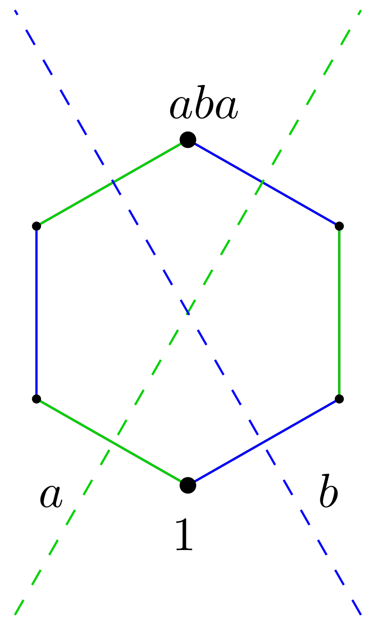
Ex 1: If $G = \mathrm{Sym}_3$, $S = \{a = (1,2), b = (2,3)\}$ and $g = (1,3)$, then $\mathcal{R} = \{aba, bab\}$ and

$$G_g = \langle a, b \mid aba = bab \rangle \cong \mathrm{Braid}_3.$$

Ex 2: If $G = \mathrm{Sym}_3$, $S = \{a = (1,2), b = (2,3), c = (1,3)\}$ and g is $(1,2,3)$, then $\mathcal{R} = \{ab, bc, ca\}$ and

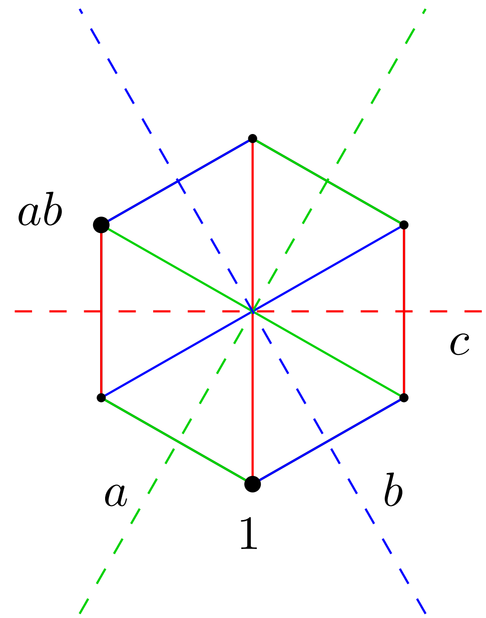
$$G_g = \langle a, b, c \mid ab = bc = ca \rangle \cong \mathrm{Braid}_3.$$

Example 1 generalized



When W is a finite Coxeter group, S is a standard Coxeter generating set, and g is the "longest element" in W, then W_g is the corresponding Artin group of finite type. The intervals are the 1-skeleton of the W-permutahedron.

Example 2 generalized



When W is a finite Coxeter group, S is the set of *all* reflections, and g is a Coxeter element, then W_g is the corresponding Artin group. The intervals are the generalized noncrossing partition lattices NC_W [Bessis, Brady-Watt].

Intervals $\Rightarrow$ Complexes

There is a natural complex K_g constructed as a quotient of the order complex of the poset $P = [1, g]$.

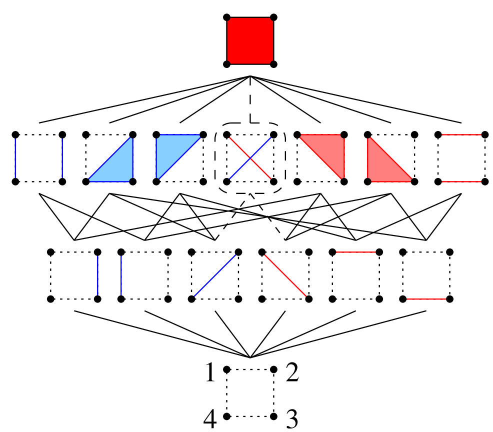
The edges in the geometric realization of P have orientations from the poset order and G-labels from the fact that P is a portion of the Cayley graph of G with respect to S.

The quotient we want is the one in which simplices are identified whenever we can do so respecting edge orientations and edge labels.

The result is a one-vertex complex K_g with $\pi_1(K_g) = G_g$.

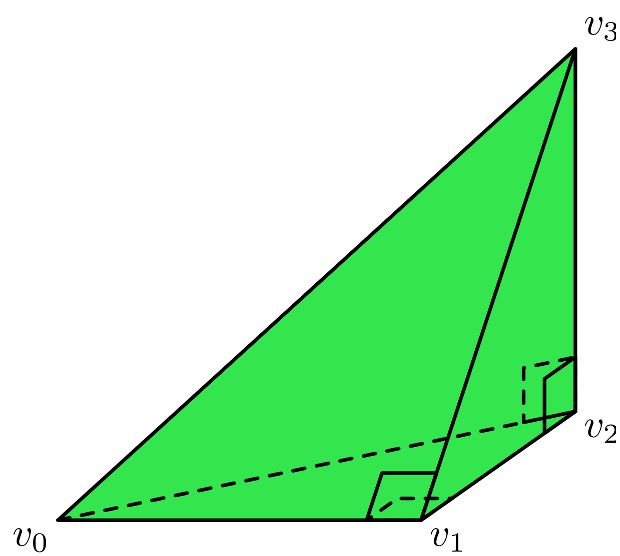
Tom's complex

Tom's complex for the braid groups is the complex K_g that results from Example 2. The group G is the symmetric group generated by all transpositions and we pull it apart at an n-cycle g. The interval $[1, g]$, as mentioned earlier, is the lattice of noncrossing partitions NC_n. Because $[1, g]$ is a "balanced" lattice, K_g is a classifying space for the braid groups Braid$_n$ [Brady].



III. Orthoschemes and complexes

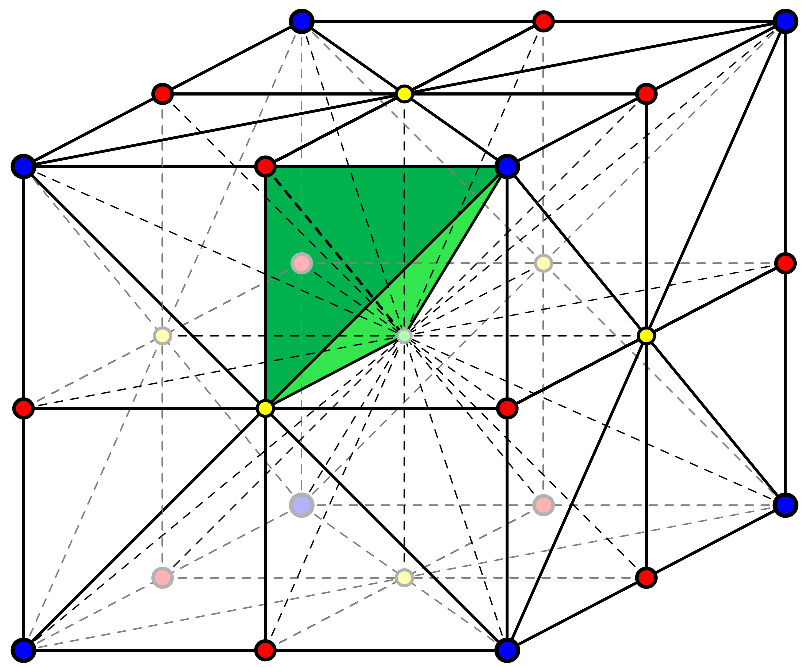
Def: An *orthoscheme* $O(v_0, v_1, \ldots, v_n)$ is the convex hull of a piecewise linear path that proceeds along mutually orthogonal directions $u_i = v_i - v_{i-1}$.



Coxeter was interested in orthoschemes because they arise when regular polytopes are metrically barycentrically subdivided.

Unit orthoschemes

Def: A *unit orthoscheme* is one where the vectors u_i are orthonormal. These metric simplices arise in the barycentric subdivision of the n-cube of side length 2.



Orthoscheme complexes

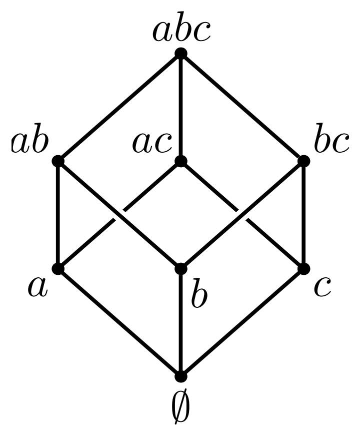
Def: If every maximal chain from the bottom to the top of a bounded poset has the same length, then this common number is called its *rank*. If every interval has a rank then P is *graded*.

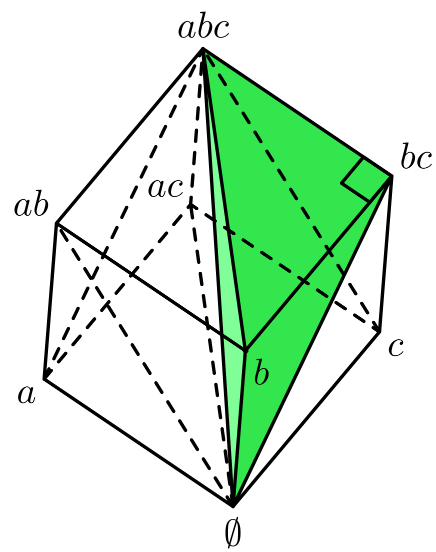
Def: The order complex of a graded poset can be turned into a piecewise Euclidean complex by turning each simplex into an orthoscheme. In particular, we make the edge corresponding to $x < y$ an edge of length $\sqrt{k}$ where k is the rank of the poset interval $P(x, y)$. The result is the *orthoscheme complex* $|P|$.

Rem: Poset products lead to orthoscheme complexes that are metric products: $|P \times Q| = |P| \times |Q|$.

Example: Boolean lattices

Def: A rank n *boolean lattice* is poset of subsets of $[n]$ under inclusion.





The orthoscheme complex of a rank n boolean lattice is a subdivided n-cube. This is a consequence of poset products leading to metric products.

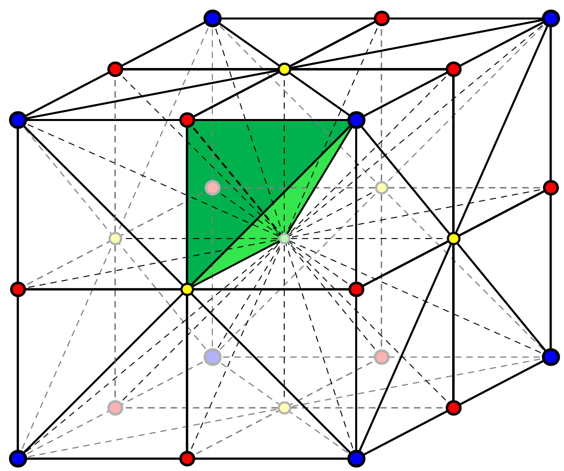
Example: Cube complexes

Every regular cell complex has a *face poset* and every poset has an order complex. These operations are almost inverses of each other in that the order complex of the face poset of a regular cell complex is homeomorphic to the original complex but it has the cell structure of its barycentric subdivision.

Observation: If X is a *cube complex* where the cubes have side length 2, then the orthoscheme complex of its face poset is isometric to the original complex but with the cell structure of its barycentric subdivision. Since we know which cube complexes are NPC, we can reformulate these conditions as conditions on their face posets.

Endpoints and diagonals

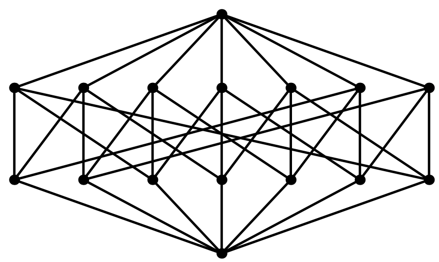
The vertices v_0 and v_n are the *endpoints* of the orthoscheme $O(v_0, \ldots, v_n)$ and the edge connecting them is its *diagonal*.

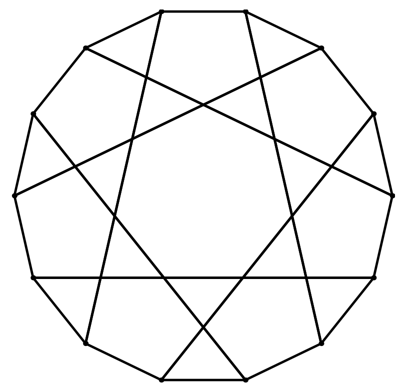


The link of the endpoint of the unit n-orthoscheme is a Coxeter simplex of type B_n. The link of its diagonal is a Coxeter simplex of type A_{n-1}.

Example: Linear subspace posets

Def: Let $L_n(\mathbb{F})$ be the poset of linear subspaces of the vector space $\mathbb{F}^n$ under inclusion.





Chains in $L_n(\mathbb{F})$ are flags, the diagonal link of the orthoscheme complex of $L_n(\mathbb{F})$ is a spherical building of type A_{n-1} and the orthoscheme metric is the one that produces the correct metric on this diagonal link. The poset $L_3(\mathbb{F}_2)$ and its diagonal link are shown.

Posets and curvature

Question: Which bounded graded posets have NPC orthoscheme complexes?

I know some aspects of the answer.

First, this collection of posets is closed under taking intervals. Next, for a fixed rank n, the entire collection is defined by the exclusion of a finite list of configurations. Finally, this list is computable in theory but not (yet) in practice - except for very small ranks.

The hope is that the eventual list is relatively easy to describe as in Gromov's link condition.

Links and joins

The orthogonality embedded in the definition of an orthoscheme means that the links of its faces decompose into spherical joins.

Def: If K and L are spherical polytopes that are vertex links of Euclidean polytopes P and Q then the *spherical join* $K * L$ is defined to be the corresponding vertex link in $P \times Q$.

Rem: The link of a vertex in an unit orthoscheme is a spherical join of two endpoint links of unit suborthoschemes. The link of an edge in a unit orthoscheme is a spherical join of two endpoint links and a diagonal link of unit suborthoschemes.

Lem: The links of simplices in a unit orthoscheme are spherical joins of spherical polytopes of type A and B.

Links and curvature

Lem: The link of a simplex in an orthoscheme complex of a bounded graded poset P is a spherical join of endpoint and diagonal links of the orthoscheme complexes of subintervals of P

Lem: Spherical joins contains short geodesic loops iff one factor contains a short geodesic loop and endpoint links never contain short geodesic loops.

Thm(Brady-M) The orthoscheme complex of a bounded graded poset is NPC iff its local diagonal links have no short geodesic loops.

In other words, it fails to be NPC iff it contains an interval whose diagonal link contains a short geodesic loop.

Diagonal Links

The diagonal link of an orthoscheme complex is the order complex of $P \setminus \{0,1\}$ (its bounding elements) with a PS metric of type A.

The vertices of the diagonal link corresponds to the elements of $P \setminus \{0,1\}$. The edges of the diagonal link indicate compatibility: there is one edge for each $x < y$.

Let $i + j + k = n$ where n is the rank of P. The metric on a maximal chain in the diagonal link of P is a spherical simplex where the edge connecting the vertex of rank i with the vertex of corank k has length θ where

$$\cos(\theta) = \sqrt{\tfrac{i}{i+j} \cdot \tfrac{k}{j+k}}.$$

Spindles and local geodesics

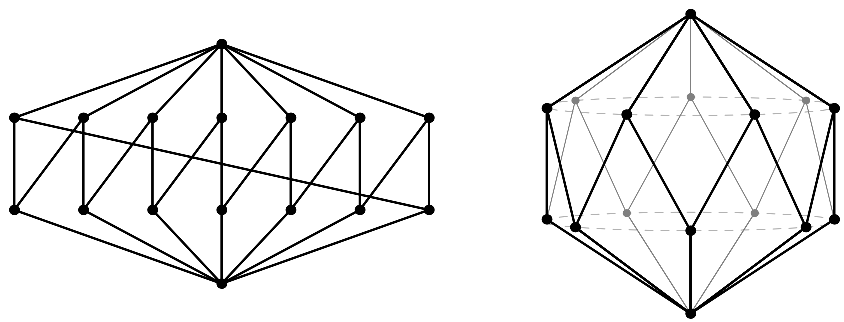
Lem: Every locally geodesic path that remains in the 1-skeleton of the diagonal link of $|P|$ is described by a spindle in P.

Three types of loops and their relations.

$$\left\{ \begin{array}{c} \text{Local geodesics} \\ \text{in a local} \\ \text{diagonal link} \end{array} \right\} \supset \left\{ \begin{array}{c} \text{Local geodesics} \\ \text{that remain in} \\ \text{its 1-skeleton} \end{array} \right\} \subset \left\{ \begin{array}{c} \text{Loops that} \\ \text{correspond} \\ \text{to spindles} \end{array} \right\}$$

Spindles

Def: Two elements in a bounded poset are *complements* if they have no nontrivial upper or lower bounds. A *spindle* is a zig-zag path in a poset where x_{i-1} and x_{i+1} are complements in the subinterval $[0, x_i]$ or $[x_i, 1]$. The length of a spindle is the sum of its edge lengths and it is short if it has length less than 2π. Here are two views of a spindle of girth 14.



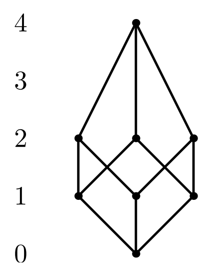
Question: Have spindles already arisen in combinatorics under another name?

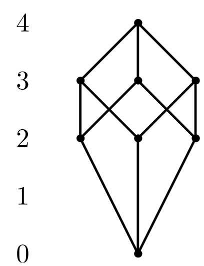
Poset curvature conjecture

Conjecture: The orthoscheme complex of a bounded graded poset is NPC iff it has no short spindles.

Thm(T.Brady-M): True for posets of rank at most 4.

The proof uses earlier work with Murray Elder. After several reductions the only short spindles in rank 4 are those shown:





4-generator Coxeter groups and Artin groups

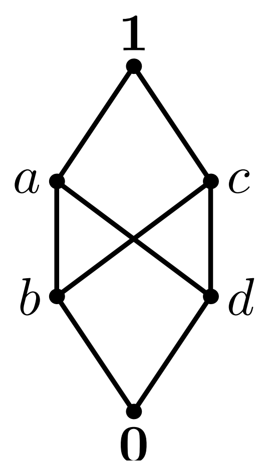
As a result Tom's complex for Braid_5 is nonpositively curved under the orthoscheme metric.

Thm: The noncrossing partition lattices of type A_4 and B_4 contain no short spindles. As a consequence their orthoscheme complexes are NPC and the A_4 and B_4 Artin groups have good curvature properties. The noncrossing partition lattices of type D_4, F_4 and H_4 do contain short spindles.

In fact, Woonjung Choi showed in her dissertation that the D_4, F_4 and H_4 complexes do not support any NPC PE metric. The D_4, F_4 and H_4 examples as illustrations of the converse of the good-true-beautiful metatheorem. As Coxeter theorists well know, type D and the exceptional types are often harder to deal with than type A and type B.

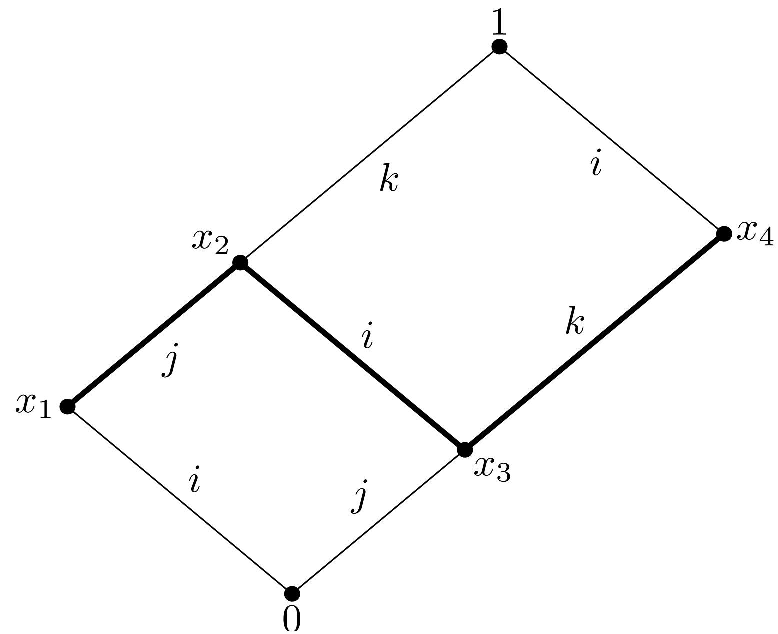
Lattices

Lem: Every spindle of girth 4 is short and a bounded graded poset contains a girth 4 spindle iff it is not a lattice.



Rem: In boolean lattices complements are unique and every spindle has girth 6 and length 2π. Thus boolean lattices have no short spindles.

Modular lattices



Prop: If P is a modular lattice then P has no short spindles.

Conj: If P is a modular lattice then its orthoscheme complex $|P|$ is NPC.

Partitions and Buildings

One reason for believing that Tom's complex is NPC in general (i.e. the orthoscheme complex for NC_n is NPC) is that for every n and for every field $\mathbb{F}$ we have the following inclusions:

$$NC_n \subset \Pi_n \subset L_n(\mathbb{F})$$

The middle poset is the full partition lattice and for the second inclusion we use the blocks to indicate which coordinates must be equal. The diagonal link of $L_n(\mathbb{F})$ is a spherical building. Because every chain in NC_n and Π_n is part of a Boolean lattice subposet, their diagonal links can be viewed as unions of apartments from this building and which apartments can be explicitly described.

Apartments in NC_n

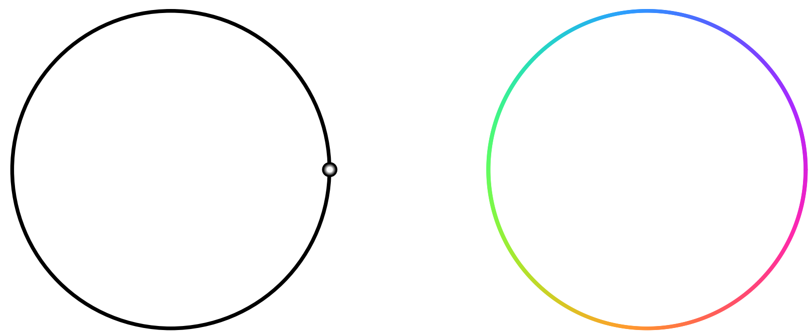
Rem: The subposets of NC_n that correspond to apartments in its diagonal link are indexed by noncrossing planar trees.

Consider a boolean subposet of NC_n. To find the corresponding tree note that the rank 1 elements correspond to edges in the n-gon. Because their join is rank 2, the edges do not cross. Because of the heights of the other joins, there can be no cycles among the edges and their union must be a tree.

The boolean subposet come from selecting a subset of the edges and taking convex hulls of the connected components.

IV. Buildings and continuous braids

Suppose we consider continuous groups as GGT objects



The Lie group $O(2,\mathbb{R})$ as a space is the union of two circles. The reflections are shown on the right, the rotations are on the left and the identity is marked. We use the reflections as our generating set and the colors are used to distinguish them. What does its Cayley graph look like?

Cayley graph of $O(2,\mathbb{R})$

The Cayley graph of $O(2,\mathbb{R})$ with respect to the set $\mathcal{R}$ of all reflections has:

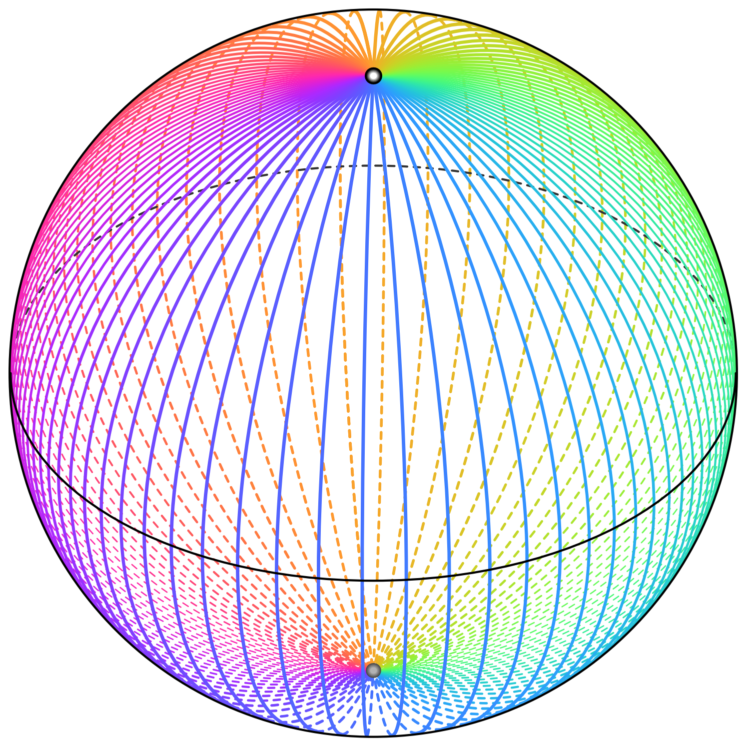
- Vertices correspond to $\mathbb{S}^1 \cup \mathbb{S}^1$ with the discrete topology.

- Edges create a complete bipartite graph: there is a unique edge (with a unique color) connecting each vertex on one circle is to each vertex on the other circle.

The Cayley graph thus looks like $\mathbb{S}^1 * \mathbb{S}^1 \cong \mathbb{S}^3$ but with a strange metric turning it into a simplicial graph.

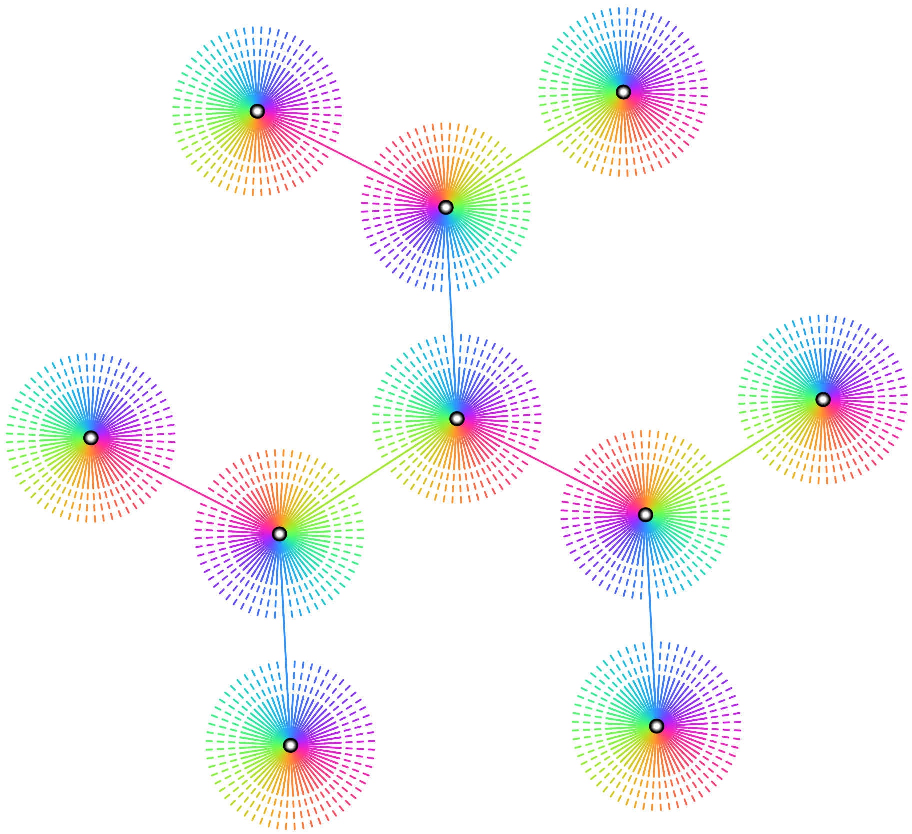
The orthogonal groups can be viewed as continuous Coxeter groups. What I'm defining is the continuous Artin group analog.

The Longitude Metric on $\mathbb{S}^2$



This is a simplicial graph, its fundamental group is free and it has a very pretty universal cover.

Its universal cover

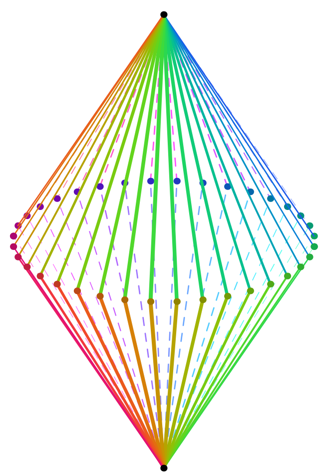


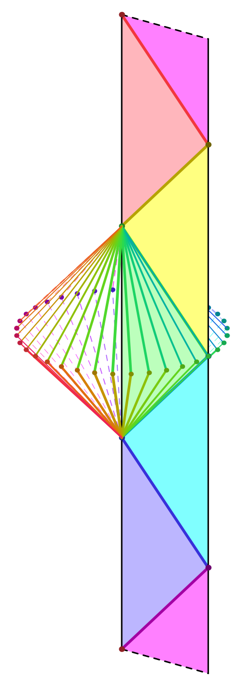
Call this circle-branching simplicial tree $T_{\mathbb{S}^1}$.

Pulling apart $O(2, \mathbb{R})$

Apply the pulling apart construction where $G = O(2, \mathbb{R})$, S is the set of all reflections, and g is a non-trivial rotation. Its factorization poset P and a portion of $\widetilde{K}$ are shown below.

$P =$

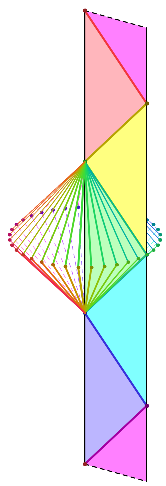




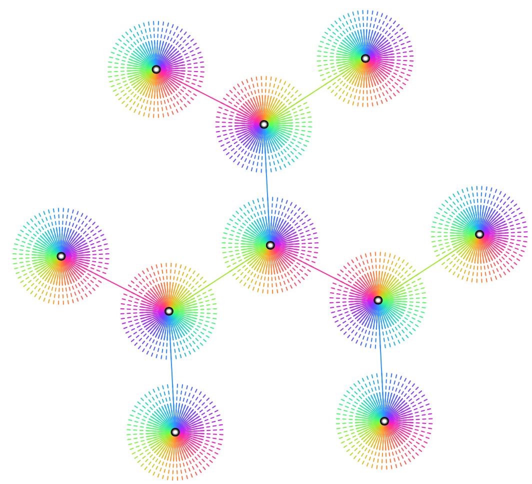
$\subset \widetilde{K}$

Structure of pulled apart $O(2,\mathbb{R})$

Prop: When $G = O(2,\mathbb{R})$ is pulled apart at a rotation g, the complex $\widetilde{K}$ is a metric product $T_{\mathbb{S}^1} \times \mathbb{R}$.



$\rightsquigarrow$



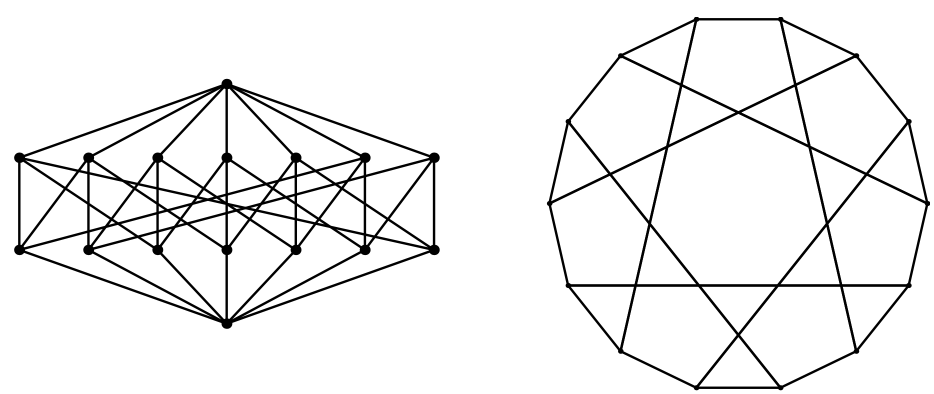
$\times \quad \mathbb{R}$

The group structure depends on the choice of g. When the rotation is rational, G_g has a non-trivial center. When it is irrational, G_g is centerless.

Factorizations in $O(n,\mathbb{R})$

Thm(T.Brady-Watt): If g is fixed-point free isometry of $\mathbb{S}^{n-1}$ then its poset P of minimal length factorizations is isomorphic to the linear subspace poset $L_n(\mathbb{R})$. Moreover, the isomorphism is defined by sending $h \in P$ to the orthogonal complement of its fixed subspace.

Notice that the poset structure is independent of g. What changes is the way edges in P are labeled by elements of $O(n,\mathbb{R})$.



Structure of pulled apart $O(n, \mathbb{R})$

Thm (M): For every fixed-point free isometry g in $O(n, \mathbb{R})$, the group G_g obtained by pulling $G = O(n, \mathbb{R})$ apart at g (with respect to all reflections) has a finite-dimensional classifying space whose universal cover is isometric to an $\widetilde{A}_{n-1}$-building cross the reals (hence CAT(0)). In addition, it has a continuous Garside structure and thus the word problem is "decidable".

The proof is essentially an application of traditional Garside constructions to the orthogonal groups, plus the factorization structure found by Brady and Watt. I call this pulled apart orthogonal group the group of continuous braids.

V. Newton's 13 spheres

The 13 sphere theorem (conjectured by Newton in conversation with David Gregory) says that 13 unit spheres cannot simultaneously touch a common unit sphere. John Leech gave a 2 page proof of this theorem in 1956. Leech reduces it to the problem of excluding a 13 vertex graph with all but one vertex degree 5 and the final vertex degree 4 and all but one region triangular with the final region a square. He then states

> *Despite the consistency of these numbers, there is in fact no polyhedron which has them. I know of no better proof of this than sheer trial.*

Using orbifolds, one can prove this without trial and error and also solve the more general challenge problem.

Spherical orbifolds

A topological disc with a local unit sphere metric has an essentially unique local isometry to $\mathbf{S}^2$. The proof is like analytic continuation.

Thm: Let X be a metric space that is topologically a 2-sphere that locally has a unit sphere metric everywhere except at 2 points (called cone points).

1. the amount of angle at both cone points is the same
2. the distance between the cone points is at most π
3. if less than π then the cone angle is $2\pi k$ for $k \in \mathbb{Z}$.

Proof: Cut open along a geodesic and map to $\mathbf{S}^2$.

Challenge problem

Let Γ be an n-vertex planar graph in which all but one vertex has degree 5 and every bounded region is a triangle. Let k be the degree of the exceptional vertex and let the unbounded region be an ℓ-gon.

1. What are the possible values for n? $n = 12m$
2. What are the possible values for k? $k = 5m$
3. What are the possible values for ℓ? $\ell = 3m$
4. What is the relationship between n, k and ℓ? Same m!
5. Classify *all* such graphs.

There are exactly 4 such graphs for each $m > 1$ and each is a branched cover of the icosahedron 1-skeleton. The 4 comes from the number of ways of selecting a vertex and a face of the icosahedron up to isomorphism.

Reference

T. Brady and J. McCammond, "Braids, posets and orthoschemes"
`arXiv:0909.4778`

Thank You!