Diagonal ideal of $(\mathbb{C}^2)^n$ and q, t-Catalan numbers

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Abstract

Let I_n be the (big) diagonal ideal of $(\mathbb{C}^2)^n$. Haiman proved that the q, t-Catalan number is the Hilbert series of a graded vector space $M_n = \bigoplus_{d_1,d_2}(M_n)_{d_1,d_2}$ spanned by a minimal set of generators for I_n. We give simple upper bounds on $\dim\ (M_n)_{d_1,d_2}$ in terms of partition numbers, and find all bi-degrees (d_1, d_2) such that $\dim(M_n)_{d_1,d_2}$ achieve the upper bounds. For such bi-degrees, we also find explicit bases for $(M_n)_{d_1,d_2}$.

q, t-Catalan numbers

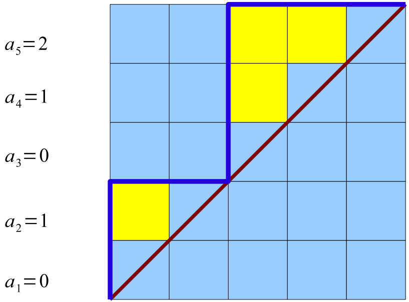
The q, t-Catalan number $C_n(q, t)$ can be defined using Dyck paths: Take the $n \times n$ square whose southwest corner is $(0, 0)$ and northeast corner is (n, n). Let $\mathcal{D}_n$ be the collection of Dyck paths, i.e. lattice paths from $(0, 0)$ to (n, n) that proceed by NORTH or EAST steps and never go below the diagonal. For any Dyck path Π, let $a_i(\Pi)$ be the number of squares in the i-th row that lie in the region bounded by Π and the diagonal. A.M.Garsia and J.Haglund showed that

$$C_n(q, t) = \sum_{\Pi \in \mathcal{D}_n} q^{\mathrm{area}(\Pi)} t^{\mathrm{dinv}(\Pi)},$$

where

$$\begin{aligned}
\mathrm{area}(\Pi) &= \sum a_i(\Pi), \\
\mathrm{dinv}(\Pi) &:= |\{(i,j) \mid i < j \text{ and } a_i(\Pi) = a_j(\Pi)\}| \\
&\qquad + |\{(i,j) \mid i < j \text{ and } a_i(\Pi) + 1 = a_j(\Pi)\}|.
\end{aligned}$$

q, t-Catalan numbers: an example



In the above example, the blue curve is a Dyck path Π,

$$\text{area}(\Pi) = 0 + 1 + 0 + 1 + 2 = 4$$
$$\text{dinv}(\Pi) = 2 + 5 = 7.$$

So this path contributes a monomial $q^4 t^7$ to the q, t-Catalan number $C_5(q, t)$.

A combinatorial characterization of q,t-Catalan numbers

Let $\mathfrak{D}_n^{catalan}$ be the set consisting of $D \subset \mathbb{N} \times \mathbb{N}$, where D contains n points satisfying the following conditions.
(a) If $(p,0) \in D$ then $(i,0) \in D, \forall i \in [0,p]$.
(b) For any $p \in \mathbb{N}$,

$$\#\{j \,|\, (p+1,j) \in D\} + \#\{j \,|\, (p,j) \in D\} \geq \max\{j \,|\, (p,j) \in D\} + 1.$$

We found the following

Proposition

The coefficient of $q^{d_1} t^{d_2}$ in the q,t-Catalan number $C_n(q,t)$ is equal to

$$\#\{D \in \mathfrak{D}_n^{catalan} \mid \deg_x D = d_1, \deg_y D = d_2\},$$

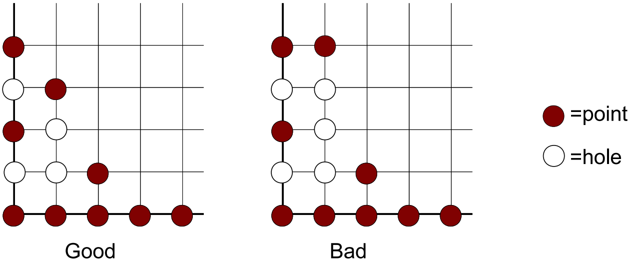
where $\deg_x D$ (resp. $\deg_y D$) is the sum of the first (resp. second) components of the n points in D.

Note: this proposition was discovered independently by A. Woo.

An example for the combinatorial characterization

The two conditions are easy to describe by picture:
(a) The bottom row has no holes.
(b) The number of holes in a column is not greater than the number of points in the next column.

In the two 9-tuples of points below, only the left one belongs to $\mathfrak{D}_9^{catalan}$.



n-tuples of points and alternating polynomials

Let $\mathfrak{D}_n$ be the set containing all the n-tuples

$$D = \{(\alpha_1, \beta_1), ..., (\alpha_n, \beta_n)\} \subset \mathbb{N} \times \mathbb{N}.$$

For any $D \in \mathfrak{D}_n$, define

$$\Delta(D) := \det \begin{bmatrix} x_1^{\alpha_1} y_1^{\beta_1} & x_1^{\alpha_2} y_1^{\beta_2} & \dots & x_1^{\alpha_n} y_1^{\beta_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n^{\alpha_1} y_n^{\beta_1} & x_n^{\alpha_2} y_n^{\beta_2} & \dots & x_n^{\alpha_n} y_n^{\beta_n} \end{bmatrix}$$

Because of alternating property of determinants with respect to rows, the polynomial $\Delta(D)$ are alternating polynomials, i.e. they satisfy the alternating condition:

$$\sigma(f) = \mathrm{sgn}(\sigma) f, \forall \sigma \in S_n.$$

It is easy to see that $\{\Delta(D)\}_{D \in \mathfrak{D}_n}$ forms a basis for the vector space of alternating polynomials.

Haiman's theorem

Haiman proves that

$$\bigcap_{1 \le i < j \le n} (x_i - x_j, y_i - y_j) = \text{ideal generated by } \Delta(D)\text{'s.}$$

Call the above ideal the **diagonal ideal** and denote it by I_n. The number of minimal generators of I_n, which is the same as the dimension of the vector space $M_n = I_n/(\mathbf{x}, \mathbf{y})I_n$, is equal to the n-th Catalan number. The space M_n is doubly graded as $\oplus_{d_1,d_2}(M_n)_{d_1,d_2}$. The q, t-Catalan number can be equivalently defined as

$$C_n(q,t) = \sum_{d_1,d_2} \dim(M_n)_{d_1,d_2} q^{d_1} t^{d_2}.$$

Question that we are interested

Question

Given a bi-degree (d_1, d_2), is there a combinatorially significant construction of the basis of $(M_n)_{d_1,d_2}$?

Using Haiman's theorem, the study of the above question is closely related to the study of q,t-Catalan numbers. The next theorem answers the question for certain bi-degrees.

Main result

Theorem

Let d_1, d_2 be non-negative integers d_1, d_2 with $d_1 + d_2 \leq \binom{n}{2}$. Define $k = \binom{n}{2} - d_1 - d_2$ and $\delta = \min(d_1, d_2)$. Then the coefficient of $q^{d_1} t^{d_2}$ in $C_n(q,t)$, which is $\dim(M_n)_{d_1,d_2}$, is less than or equal to $p(\delta, k)$, and the equality holds if and only if one the following conditions holds:

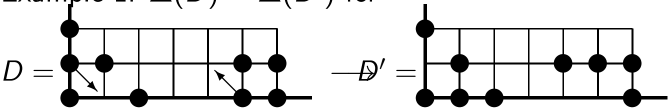
- $k \leq n - 3$, *or*
- $k = n - 2$ *and* $\delta = 1$, *or*
- $\delta = 0$.

In case the equality holds, there is an explicit construction of a basis of $(M_n)_{d_1,d_2}$.

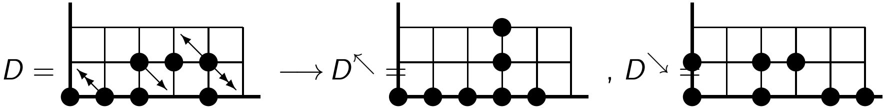
Step I of the proof: asymptotic behavior

Let $\overline{\Delta D}$ be the image of ΔD in M_n.
For n sufficiently large, we observed certain linear relations among $\overline{\Delta(D)}$ which are combinatorially simple and essential for the construction of a basis for $(M_n)_{d_1,d_2}$.

Example 1: $\overline{\Delta(D)} = \overline{\Delta(D')}$ for



Example 2: $2\overline{\Delta(D)} = \overline{\Delta(D^{\nwarrow})} + \overline{\Delta(D^{\searrow})}$ for



Step II of the proof: construct the map φ

We define a map φ sending an alternating polynomial f into the polynomial ring

$$\mathbb{C}[\rho] := \mathbb{C}[\rho_1, \rho_2, \rho_3, \dots].$$

The map has two desirable properties: (i) for many f, $\varphi(f)$ can be easily computed, and (ii) for each bi-degree (d_1, d_2), φ induces a morphism $\bar{\varphi} : (M_n)_{d_1,d_2} \to \mathbb{C}[\rho]$, and the linear dependency is easier to check in $\mathbb{C}[\rho]$ than in $(M_n)_{d_1,d_2}$. Then we explicitly construct n-tuples of points D's, such that the image $\varphi(\Delta(D))$'s are linearly independent as polynomials in $\mathbb{C}[\rho]$. $\square$

Conclusion

- The study of the bi-graded module M_n provides new insight to the study of the q, t-Catalan numbers.
- The map φ naturally arises in the study of M_n, and may be useful in the study of the geometry of the Hilbert schemes of points.

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