Toric Ideals of Flow Polytopes

Polytopes

Flow polytopes
- $G = (V, A)$ directed graph
- $\mathbf{d} \in \mathbb{Z}^V$ demand vector
- $\mathbf{l}, \mathbf{u} \in \mathbb{Z}^A_{\geq 0}$ lower and upper bounds

$$F_G(\mathbf{d}, \mathbf{u}, \mathbf{l}) = \left\{ \mathbf{f} \in \mathbb{R}^A_{\geq 0} \ : \ \mathbf{l} \leq \mathbf{f} \leq \mathbf{u}, \sum_{\mathrm{head}(a)=v} f_a - \sum_{\mathrm{tail}(a)=v} f_a = d_v \ \text{ for all } v \in V \right\}$$

Transportation polytopes
- m, n number of rows and columns
- $\mathbf{r} \in \mathbb{N}^m$ row sums
- $\mathbf{c} \in \mathbb{N}^n$ column sums

$$T_{\mathbf{rc}} = \left\{ A \in \mathbb{R}^{m \times n}_{\geq 0} \ : \ \sum_{i=1}^{m} a_{ij} = c_j, \ \sum_{j=1}^{n} a_{ij} = r_i \right\}$$

Birkhoff polytopes
- size $n \in \mathbb{N}$

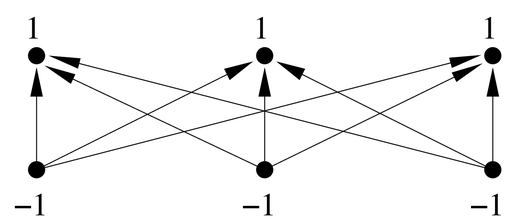
$$B_n = \mathrm{conv}\{(n \times n)\text{-permutation matrices}\} = T_{(1,\dots,1)(1,\dots,1)}$$

Remark
- Birkhoff polytopes are transportation polytopes.
- Transportation polytopes are flow polytopes.

Example
- $B_3 = T_{(1,1,1)(1,1,1)}$

$$B_3 \cap \mathbb{Z}^{3\times 3} = \left\{ \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix}, \begin{bmatrix} 0&0&1\\1&0&0\\0&1&0 \end{bmatrix}, \begin{bmatrix} 0&1&0\\0&0&1\\1&0&0 \end{bmatrix}, \begin{bmatrix} 1&0&0\\0&0&1\\0&1&0 \end{bmatrix}, \begin{bmatrix} 0&0&1\\0&1&0\\1&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&0\\1&0&0\\0&0&1 \end{bmatrix} \right\}$$

B_3 as a flow polytope:
$G = \vec{K}_{33}$, $\mathbf{u} = \mathbf{1}$, $\mathbf{l} = \mathbf{0}$
Here: integer flows $\leftrightarrow$ matchings



Toric Ideals

Definition of toric ideals
- $\mathcal{A} \subseteq \mathbb{Z}^d$ finite set
- Typical choice: P lattice polytope, $\mathcal{A} = P \cap \mathbb{Z}^d$ = lattice points in P

$$I_{\mathcal{A}} = \left\langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \ : \ \sum_{\mathbf{a}\in\mathcal{A}} u_{\mathbf{a}}\mathbf{a} - \sum_{\mathbf{a}\in\mathcal{A}} v_{\mathbf{a}}\mathbf{a} = \mathbf{0}, \ \ \mathbf{u}, \mathbf{v} \in \mathbb{N}^{\mathcal{A}} \right\rangle$$

Example
- $I_{B_3} = \langle x_{()}x_{(123)}x_{(132)} - x_{(12)}x_{(13)}x_{(23)} \rangle$
  The variables are indexed by permutations in cycle notation.

Main Result

Theorem: Toric ideals of flow polytopes are generated in degree 3.

$\Downarrow$ implies

Conjecture (Diaconis and Eriksson, [1]): Toric ideals of Birkhoff polytopes are generated in degree 3.

Applications
- Algebraic statistics:
  - Generating sets of toric ideals can be used for sampling from certain distributions, [2] $\rightsquigarrow$ Diaconis-Eriksson conjecture.
- Algebraic geometry (toric varieties):
  - P smooth $\leftrightarrow$ edge directions at every vertex of P form a lattice basis $\leftrightarrow$ toric variety X_P smooth
  - *Conjecture:* If P is a smooth lattice polytope, its toric ideal is generated by quadratic binomials, [3].

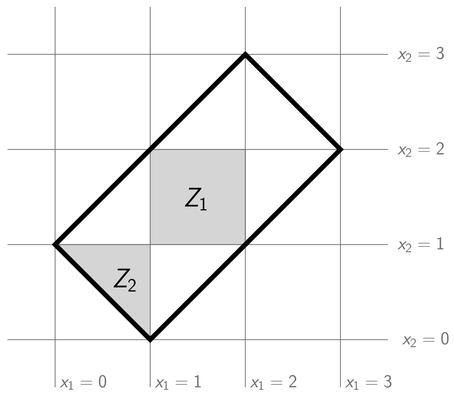
Results on Gröbner bases

- Revlex Gröbner bases of I_{B_n} have at most degree n, [2].
- This bound is tight for some revlex orders.
- Similar statements hold for (smooth) transportation polytopes.

Hyperplane Subdivisions

Cell decomposition of flow polytopes:

- Subdivide polytope into cells along hyperplanes of type $\{\mathbf{x} : x_a = k\}$ for $k \in \mathbb{Z}$.
- Cells of flow polytopes are flow polytopes.

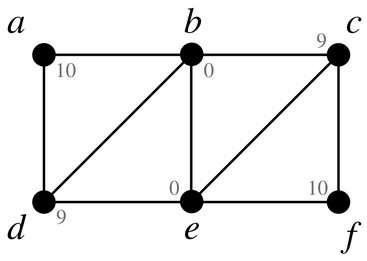


Hyperplane subdivision method [4]
- *It is sufficient to prove the Main Theorem for cells of flow polytopes.*
- This follows from the following correspondence, [5]:

  Let Δ be a regular, unimodular triangulation of P. Then, minimal non-faces of $\Delta \longleftrightarrow$ initial terms of Gröbner basis of I_P

  Example: A regular, unimodular triangulation and the corresponding Gröbner basis:

$$\mathcal{G} = \{ac - b^2, ae - bd, af - be, bf - ce, cd - be, df - e^2\}$$



Proof of the Main Theorem (idea)

Transportation polytopes are sufficient
- The general case can be reduced to this special case.

Setting:
- Z full dimensional cell of $T_{\mathbf{rc}}$
- $M, N \in Z \quad \rightsquigarrow \quad d(M, N) = \#$ of entries that are different (Hamming distance)
- $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_Z \quad \rightsquigarrow \quad d'(\mathbf{u}, \mathbf{v}) = \min\{d(M, N) \ : \ M \in \mathrm{supp}(\mathbf{u}), \ N \in \mathrm{supp}(\mathbf{v})\}$

- Suppose that the theorem is wrong, i. e. there exists a binomial $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I_Z$ of degree $k \geq 4$ that cannot be expressed by binomials in I_Z of smaller degree.
- Choose $\mathbf{u}$ and $\mathbf{v}$ s. t. $t = d'(\mathbf{u}, \mathbf{v})$ is minimal over all binomials with this property.
- Let $M^1 \in \mathrm{supp}(\mathbf{u})$ and $N^1 \in \mathrm{supp}(\mathbf{v})$ be matrices with $d(M^1, N^1) = t$.
- $t \geq 4$ as row and column sums in M^1 and N^1 agree.

Lemma:
- It is impossible that $\begin{bmatrix} 1&0\\0&1 \end{bmatrix}$ is a submatrix of M^1 and $\begin{bmatrix} 0&1\\1&0 \end{bmatrix}$ is a submatrix of N^1 and both are obtained by choosing the same rows and columns.
- This statement remains true if we replace in one of the two submatrices a 0 by a 1 or a 1 by a 0.

- As $M^1 \neq N^1$, we may assume:

$$M^1 = \begin{bmatrix} 1 & \dots & \dots \\ 0 & \dots & \dots \\ & \dots & \dots \end{bmatrix} \qquad N^1 = \begin{bmatrix} 0 & \dots & \dots \\ 1 & \dots & \dots \\ & \dots & \dots \end{bmatrix}$$

- Using the lemma and the fact that row sums in both matrices are equal, we obtain:

$$M^1 = \begin{bmatrix} 1 & 0 & \dots \\ 0 & 1 & \dots \\ \dots & & \dots \end{bmatrix} \qquad N^1 = \begin{bmatrix} 0 & 1 & \dots \\ 1 & 0 & \dots \\ \dots & & \dots \end{bmatrix}$$

- Applying the lemma again, we obtain:

$$M^1 = \begin{bmatrix} 1 & 1 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ & \dots & & \dots \end{bmatrix} \qquad N^1 = \begin{bmatrix} 0 & 0 & 1 & \dots \\ 1 & 0 & 1 & \dots \\ & \dots & & \dots \end{bmatrix}$$

- Reapplying the same arguments, we get

$$M^1 = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & \dots \\ 0 & 1 & 0 & 1 & 0 & \dots \\ & & \dots & & & \dots \end{bmatrix} \qquad N^1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & \dots \\ 1 & 0 & 1 & 0 & 1 & \dots \\ & & \dots & & & \dots \end{bmatrix}$$

- This can be done again and again. As there are only finitely many columns, we reach a contradiction. □

References

[1] Persi Diaconis and Nicholas Eriksson. Markov bases for noncommutative Fourier analysis of ranked data. *Journal of Symbolic Computation*, 41(2):182–195, 2006.

[2] Persi Diaconis and Bernd Sturmfels. Algebraic algorithms for sampling from conditional distributions. *Annals of Statistics*, 26(1):363–397, 1998.

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[4] Christian Haase and Andreas Paffenholz. Quadratic Gröbner bases for smooth 3×3 transportation polytopes. *Journal of Algebraic Combinatorics*, 30(4):477–489, 2009.

[5] Bernd Sturmfels. *Gröbner bases and convex polytopes*, volume 8 of *University Lecture Series*. American Mathematical Society, Providence, Rhode Island, 1996.

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