Toric Ideals of Flow Polytopes

Polytopes

Flow polytopes
- \( G = (V,A) \) directed graph
- \( d \in \mathbb{Z}^V \) demand vector
- \( l, u \in \mathbb{Z}_{\geq 0}^V \) lower and upper bounds

\[ F_G(d,u,l) = \{ f \in \mathbb{R}_{\geq 0}^V : l \leq f \leq u, \sum_{v \in V} f_v - \sum_{w \in N(v)} f_w = d_v \quad \text{for all} \ v \in V \} \]

Transportation polytopes
- \( m, n \) number of rows and columns
- \( r \in \mathbb{N}^m \) row sums
- \( c \in \mathbb{N}^n \) column sums

\[ T_{rc} = \{ A \in \mathbb{R}_{\geq 0}^{m \times n} : \sum_{j=1}^m a_{ij} = r_i, \quad \sum_{i=1}^n a_{ij} = c_j \} \]

Example
- \( B_3 = T_{(1,1,1)(1,1,1)} \)

Here: integer flows \( \leftrightarrow \) matchings

Toric Ideals

Definition of toric ideals
- \( A \subseteq \mathbb{Z}^d \) finite set
- Typical choice: \( P \) lattice polytope, \( A = P \cap \mathbb{Z}^d \) lattice points in \( P \)

\[ I_A = \left\langle x^a \cdot x^b : a, b \in A, a \neq b \right\rangle \]

Example
- \( I_{B_3} = \langle x^{(1,1,1)}(1,0,0) - x^{(1,0,0)}(1,1,1) \rangle \)

The variables are indexed by permutations in cycle notation.

Main Result

Theorem: Toric ideals of flow polytopes are generated in degree 3.

\( \Downarrow \) implies

Conjecture (Diaconis and Eriksson, [1]): Toric ideals of Birkhoff polytopes are generated in degree 3.

Applications
- Generating sets of toric ideals can be used for sampling from certain distributions, [2] → Diaconis-Eriksson conjecture.
- Algebraic geometry (toric varieties)
  - \( P \) smooth → edge directions of every vertex of \( P \) form a lattice basis → toric variety \( \mathbb{X}_P \) smooth
- Conjecture: If \( P \) is a smooth lattice polytope, its toric ideal is generated by quadratic binomials, [3].

Results on Gröbner bases
- Revlex Gröbner bases of \( I_{B_3} \) have at most degree \( n \), [2].
- This bound is tight for some revlex orders.
- Similar statements hold for (smooth) transportation polytopes.

Hyperplane Subdivisions

Cell decomposition of flow polytopes:
- Subdivide along hyperplanes of type \( \{ x : x_k = k \} \) for \( k \in \mathbb{Z} \)
- Cells of flow polytopes are flow polytopes.

Hyperplane subdivision method [4]
- It is sufficient to prove the Main Theorem for cells of flow polytopes.
- This follows from the following correspondence, [5]:
  - Let \( \Delta \) be a regular, unimodular triangulation of \( P \). Then, minimal non-faces of \( \Delta \) → initial terms of Gröbner basis of \( I_P \)

Example: A regular, unimodular triangulation and the corresponding Gröbner basis:

\[ G = \langle ab - b^2, e d - c e, a f - be, b f - c e, c d - b e, d f - e^2 \rangle \]

Proof of the Main Theorem (idea)

Transportation polytopes are sufficient
- The general case can be reduced to this special case.
  - Setting:
    - \( Z \) full dimensional cell of \( T_{rc} \)
    - \( M, N \in Z \) = \( d(M,N) = \# \) of entries that are different (Hamming distance)
    - \( x^a - x^b \in \mathbb{Z}^d \) = \( d(u,w) = \min d(M,N) : M \supseteq (u,w) \in supp(v) \)
- Suppose that the theorem is wrong, i.e. there exists a binomial \( x^a - x^b \) of degree \( k = 4 \) that cannot be expressed by binomials in \( \mathbb{Z}_2 \) of smaller degree.
  - Choose \( u \) and \( v \) s.t. \( d(u,w) = \min \) over all binomials with this property.
- Let \( M^1 \subseteq supp(u) \) and \( N^1 \subseteq supp(v) \) be matrices with \( d(M^1, N^1) = t \).
  - \( t \leq 4 \) as row and column sums in \( M^1 \) and \( N^1 \) agree.

Lemma: It is impossible that \( \langle 1 \rangle \) is a submatrix of \( M^1 \) and \( \langle 0,1 \rangle \) is a submatrix of \( N^1 \) and both are obtained by choosing the same rows and columns.
  - This statement remains true if we replace in one of the two submatrices a by 1 or a 1 by 0.
  - As \( M^1 \neq N^1 \), we may assume:

* Using the lemma and the fact that row sums in both matrices are equal, we obtain:

\[ M^1 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}, \quad N^1 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix} \]

* Applying the lemma again, we obtain:

\[ M^2 = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}, \quad N^2 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix} \]

* Reapplying the same arguments, we get:

\[ M^3 = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}, \quad N^3 = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix} \]

* This can be done again and again. As there are only finitely many columns, we reach a contradiction. \( \square \)

References


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