A Closed Character Formula for Symmetric Powers of Irreducible Representations

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We prove a closed character formula for the symmetric powers $S^N V(\lambda)$ of a fixed irreducible representation $V(\lambda)$ of a complex semi-simple Lie algebra $\mathfrak{g}$ by means of partial fraction decomposition. The formula involves rational functions in rank of $\mathfrak{g}$ many variables which are easier to determine than the weight multiplicities of $S^N V(\lambda)$ themselves. We compute those rational functions in an interesting case which allows a comparison with Weyl's character formula. Furthermore, we introduce a residue-type generating function for the weight multiplicities of $S^N V(\lambda)$ and explain the connections between our character formula, vector partition functions and iterated partial fraction decomposition.

1 The Character Formula

Let

$$\begin{aligned}
\mathfrak{g} &= \text{a semi-simple complex Lie algebra of rank } r,\\
X &= \text{its weight lattice } X = \oplus_{i=1}^r \mathbb{Z}\omega_i,\\
Q &= \text{its root lattice } Q = \oplus_{i=1}^r \mathbb{Z}\alpha_i,\\
W &= \text{its Weyl group},\\
\rho &= \text{sum of fundamental weights},\\
V(\lambda) &= \text{the irreducible representation of highest weight } \lambda,\\
m_\lambda &= \text{the weight multiplicity function } m_\lambda : X \to \mathbb{N},\ m_\lambda(\nu) = \dim V(\lambda)_\nu,\\
S^N V(\lambda) &= \text{the } N\text{-th symmetric power of } V(\lambda).
\end{aligned}$$

Theorem (Character formula). Let $\mathfrak{g}$, $V(\lambda)$, m_λ be as above and set $q = e^{i\langle\cdot,x\rangle} = (q_1, \ldots, q_r)$. Then,

$$\operatorname{Char} S^N V(\lambda)(ix) = \sum_{\nu \in X} q^{N\nu} \sum_{k=1}^{m_\lambda(\nu)} A_{\nu,k}(q) \cdot p_k(N) \in \mathbb{C}(q_1, \ldots, q_r)[X] \tag{1}$$

with rational functions $A_{\nu,k}(q) \in \mathbb{C}(q_1, \ldots, q_r)$ and polynomials $p_k(N) \in \mathbb{Q}[N]$ of degree $k-1$ given by

$$p_k(N) = \binom{N+k-1}{N}. \tag{2}$$

Furthermore, for a weight $\mu \in X$ and $l = 0, \ldots, m_\lambda(\mu) - 1$ we have

$$A_{\mu, m_\lambda(\mu)-l}(q) = \frac{(-1)^l}{l! q^{l\mu}} \cdot \frac{d^l}{(dz)^l} \left[\prod_{\nu \in X \setminus \mu} \frac{1}{(1 - q^\nu z)^{m_\lambda(\nu)}} \right]_{z = q^{-\mu}}. \tag{3}$$

Sketch of proof.

1. Molien's formula as stated in [Pro07]:
$$\operatorname{Char} SV(\lambda) = \sum_{N=0}^\infty z^N \operatorname{Char} S^N V(\lambda) = \prod_{\nu \in X} \frac{1}{(1 - e^\nu z)^{m_\lambda(\nu)}}.$$
2. Partial fraction decomposition of the product expression on the right-hand side with respect to z.
3. Compare the coefficients of the resulting power series in z to the graded character of the symmetric algebra of $V(\lambda)$.

One can immediately compute the character of the symmetric powers of a multiplicity free irreducible representation. For example:

Corollary. Let $\mathfrak{g} = \mathfrak{sl}(r+1, \mathbb{C})$ and consider its fundamental representation $V(\omega_1)$. Set $\omega_0 = \omega_{r+1} = 0$. Then,

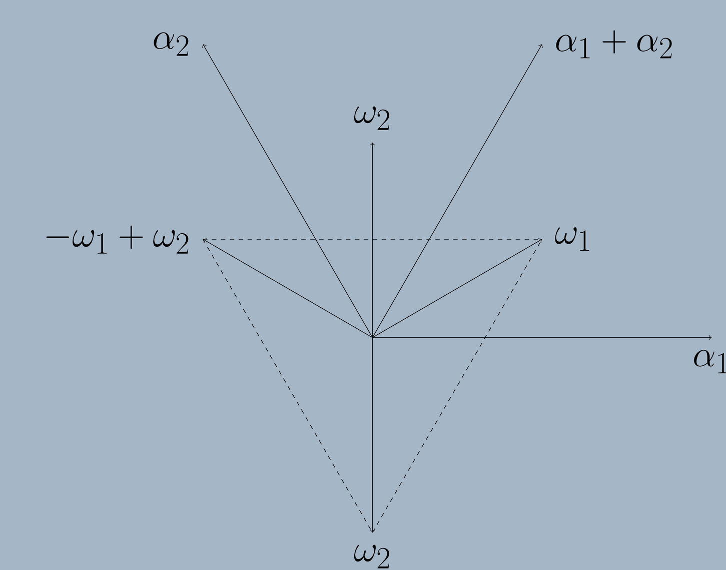
$$\operatorname{Char} S^N V(\omega_1) = \sum_{i=0}^r \frac{e^{N(-\omega_i + \omega_{i+1})}}{\prod_{\substack{j=0\\ j\neq i}}^r \left(1 - e^{-\omega_j + \omega_{j+1} - (-\omega_i + \omega_{i+1})}\right)} \tag{4}$$

$$= \sum_{i=0}^r \frac{e^{N(-\omega_i + \omega_{i+1})}}{\prod_{k=1}^i \left(1 - e^{\alpha_k + \alpha_{k+1} + \ldots + \alpha_i}\right) \prod_{k=i+1}^r \left(1 - e^{-(\alpha_{k+1} + \alpha_{k+2} + \ldots + \alpha_r)}\right)}. \tag{5}$$

Sketch of proof.

1. Equation (4) follows immediately from the Theorem.
2. Equation (5) is a straightforward computation of the expressions $-\omega_j + \omega_{j+1} - (-\omega_i + \omega_{i+1})$ in terms of the positive roots Q^+.

Example. For $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{C})$ and its fundamental representation $V(\omega_1)$ we have the following picture:



The dashed lines indicate the Weyl polytope, and the character of $S^N V(\omega_1)$ is given by

$$\operatorname{Char} S^N V(\omega_1) = \frac{e^{N\omega_1}}{\left(1 - e^{-\alpha_1}\right)\left(1 - e^{-(\alpha_1+\alpha_2)}\right)} + \frac{e^{N(-\omega_1+\omega_2)}}{\left(1 - e^{\alpha_1}\right)\left(1 - e^{-\alpha_2}\right)} + \frac{e^{N(-\omega_2)}}{\left(1 - e^{\alpha_1+\alpha_2}\right)\left(1 - e^{\alpha_2}\right)}.$$

Note that $S^N V(\omega_1) = V(N\omega_1)$ is irreducible which leads to a comparison with Weyl's character formula for irreducible representations.

2 Comparison with Weyl's Character Formula

Weyl's character formula states that for an irreducible representation $V(\lambda)$:

$$\operatorname{Char} V(\lambda) = \sum_{w \in W} \frac{(-1)^w e^{w(\lambda+\rho)-\rho}}{\prod_{\beta \in Q^+} (1 - e^{-\beta})}$$

In the case $\mathfrak{g} = \mathfrak{sl}(r+1, \mathbb{C})$ and $S^N V(\omega_1) = V(N\omega_1)$ we can compare Weyl's character formula with the expression in the previous Corollary. Weyl's formula is Weyl group invariant whereas our formula is not. But our formula has only $r+1$ summands, individually equipped with r poles, compared to $r!$ summands and (summation independent) $\frac{1}{2}r(r+1)$ poles in Weyl's expression.

Goal I. For arbitrary $\lambda \in X$ - compute an explicit expression for $\operatorname{Char} S^N V(\lambda)$ from the character formula given in the Theorem as it is done in the Corollary for $S^N V(\omega_1) = V(N\omega_1)$. This amounts to give an explicit expression of the rational functions $A_{\mu, m_\lambda(\mu)-l}(q)$ stated in the Theorem in terms of the weight multiplicity function $m_\lambda : X \to \mathbb{N}$ and (linear combinations of) roots $\alpha \in Q$.

Goal II (rather a bold Question). Is it possible to establish a Weyl group invariant character formula for $S^N V(\lambda)$ from this?

3 A Residue-Type Generating Function for the Weight Multiplicities

Proposition. Let $\mathfrak{g}$ and $V(\lambda)$ be as above. Let $m_{\lambda,N}$ be the weight multiplicity function of the N-th symmetric power $S^N V(\lambda)$ and $\mu \in X$ be a fixed weight. Then, the formal power series $\sum_{N=0}^\infty z^N m_{\lambda,N}(\mu)$ is a holomorphic function in the variable z on $|z| \leq R < 1$. Moreover, we have the identity

$$\sum_{N=0}^\infty z^N m_{\lambda,N}(\mu) = \frac{1}{(2\pi)^r} \int_{T^r} e^{-i\langle\mu,x\rangle} \prod_{\nu \in X} \frac{1}{(1 - e^{i\langle\nu,x\rangle} z)^{m_{\lambda,1}(\nu)}} dx. \tag{6}$$

Sketch of proof. The dimension of the symmetric power of a representation grows sub-exponentially in N.

With the Theorem at hand we are able to explain why the generating function in Equation (6) is of residue-type.

Corollary (Residue-type). Let $\mathfrak{g}$ and $V(\lambda)$ be as above, and $m_{\lambda,N}$ be the weight multiplicity function of the N-th symmetric power $S^N V(\lambda)$. Let $\mu \in X$ be a fixed weight and denote $q^\mu = e^{i\langle\mu,x\rangle}$ as above. Then,

$$m_{\lambda,N}(\mu) = \frac{1}{(2\pi)^r} \int_{T^r} q^{-\mu} \sum_{\nu \in X} q^{N\nu} \sum_{k=1}^{m_\lambda(\nu)} A_{\nu,k}(q) \cdot p_k(N) dx. \tag{7}$$

4 Related Research

For an integral matrix $A \in \mathbb{Z}^{(m,d)}$ with $\ker(A) \cap \mathbb{R}^d_+ = \{0\}$ we define the vector partition function $\phi_A : \mathbb{Z}^m \to \mathbb{N}$ by

$$\phi_A(b) = \#\{x \in \mathbb{N}^d : Ax = b\}.$$

Let $c_1, \ldots, c_d$ denote the columns of A and use multiexponent notation $z^b = z_1^{b_1} \cdots z_m^{b_m}$, $b \in \mathbb{Z}^m$. Then, as stated in [Bli09, Equation (1)], on $\{z \in \mathbb{C}^m : |z^{c_k}| < 1 \text{ for } k = 1, \ldots, d\}$ we have the identity

$$f_A(z) := \sum_{b \in \mathbb{Z}^m} \phi_A(b) z^b = \prod_{k=1}^d \frac{1}{1 - z^{c_k}}$$

and

$$\phi_A(b) = \text{const}\left[f_A(z) \cdot z^{-b}\right].$$

Now, there is an obvious connection between the graded character of the symmetric algebra $SV(\lambda)$ of an irreducible representation $V(\lambda)$ of a complex semi-simple Lie algebra $\mathfrak{g}$ and the theory of vector partition functions, which is given by Molien's formula. Namely, if $\mathfrak{g}$ is of rank r, then one has a matrix $A \in \mathbb{Z}^{(r+1, \dim V(\lambda))}$ encoding the weights of $V(\lambda)$ in terms of the coordinate system given by the fundamental weights $\omega_1, \ldots, \omega_r$. This information corresponds to the first r rows of each column of A. In addition to that, we have the $(r+1)$-th row which associates to the grading given by z in Molien's formula.

In contrast to the computational and algorithmic aspects of iterated partial fraction decomposition as proposed in [Bec04] and continued e.g. in [Bli09] for "arbitrary" matrices A, our interests are different. They lie in investigating further the closed character formulas for the symmetric powers.

References

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