Chain enumeration of $k$-divisible noncrossing partitions of classical types

Jang Soo Kim

University of Paris 7

Introduction

Definition (1972, Kreweras) : Noncrossing Partitions

NC(n) is the set of noncrossing partitions of $[n] = \{1, 2, \ldots, n\}$.

- Each element in a partition is called a block.
- If the size of each block of $\pi \in \text{NC}(n)$ is divisible by $k$, it is called a $k$-divisible noncrossing partition.
- For $\pi \in \text{NC}(n)$, the block size vector of $\pi$ is $(b_1, b_2, \ldots, b_n)$ where $b_i$ is the number of blocks of size $i$.
- The block size vector of the following diagram is $(4, 2, 1, 0, 1, 0, 0, \ldots)$.

Theorem (1972, Kreweras)

The number of $\pi \in \text{NC}(n)$ with the block size vector $(b_1, b_2, \ldots, b_n)$ is equal to

$$\frac{1}{n!} \sum_{r=0}^{n} \binom{n}{r} \binom{n-r}{b_1} \binom{n-b_1}{r}.$$

- For $\pi_1, \pi_2 \in \text{NC}(n)$, we can split the partial order $\pi_1 \leq \pi_2$ if $\pi_1$ is a refinement of $\pi_2$.

NC(n) is graded with rank($\pi$) = $n - \text{blk}(\pi)$.

A multichain $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_r \in \text{NC}(n)$ has rank jump vector $(r_1, r_2, \ldots, r_n)$ if

$$\begin{align*}
0 \leq r_1 \leq r_2 \leq \cdots \leq r_n, & \leq \iota \leq \iota + 1 \\
\text{or}\quad \text{rank}(r_1) \leq \text{rank}(r_2) \leq \cdots \leq \text{rank}(r_n), & \leq \iota + 1
\end{align*}$$

Theorem (1980, Edelman)

The number of multichains $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_r \in \text{NC}(n)$ with rank jump vector $(r_1, r_2, \ldots, r_n)$ is equal to

$$\frac{1}{r!} \binom{n+2r-1\iota}{r}. $$

- The number of $\pi \in \text{NC}(n)$ with $\iota$ blocks is the Narayana number $\frac{1}{\iota+1} \binom{2\iota + 1}{\iota}.$
- The number of maximal chains in $\text{NC}(n)$ is $n^{\iota-2}$ (Stanley (1996)) found a bijection between them with parking functions.

Definition (2003, Bessis; Brady and Watt): Noncrossing partitions of a Coxeter group $W$

- $(W, S)$ : finite Coxeter system, $T = \{s_1, \ldots, s_n, w \in W\}$ : the set of reflections
- $I(W) = \{w \in W : w < 1 \}$ : the set of reflections
- $\text{rank}(w) = \text{rank}(w')$ if $w < w'$
- For a Coxeter element $c$ of $W$, for example $c = s_1s_2s_1$, define $\text{NC}(W) = \{w \in W : 1 < w < c\}$

Combinatorial models for classical Coxeter groups

- Type A (1997, Reiner)
- Type B (2004, Reiner, Stanley, Reiner)
- Type D (2004, Reiner, Stanley, Reiner)

- In 2006, Armstrong defined $L$-divisible noncrossing partitions $\text{NC}^{(L)}(W)$ for each Coxeter group $W$.
- The combinatorial models $\text{NC}^{(L)}(A_n)$ and $\text{NC}^{(L)}(B_n)$ are the natural ones.
- The model $\text{NC}^{(L)}(G_2)$ was found by Müller and Krattenthaler in 2007.

Theorem (2007, Müller and Krattenthaler)

The number of multichains $\pi_1 \leq \pi_2 \leq \cdots \leq \pi_r \in \text{NC}^{(L)}(n)$ with rank jump vector $(r_1, r_2, \ldots, r_n)$ is equal to

$$\frac{1}{r!} \binom{n+2r-1\iota}{r} \binom{n-r}{b_1} \binom{n-b_1}{r}.$$

The number of such multichains in $\text{NC}^{(L)}(n)$ is $\prod_{i=1}^{\iota} \binom{k_i-1}{b_i-1}.$

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Main purpose : Prove the above theorem combinatorially!

- Müller and Krattenthaler mentioned that finding a combinatorial proof for type D seems rather hopeless.
- Since type D noncrossing partitions are special cases of type B noncrossing partitions, understanding type B noncrossing partitions is important.
- We will find a connection between $\text{NC}(n)$ and the easiest object NC(n).

Two interpretations for $\text{NC}_B(n)$

Equivalent definition of $\text{NC}_B(n)$

$\text{NC}_B(n)$ is the set of partitions $\pi$ of $[n] = \{1, 2, \ldots, n\}$ such that

- If $B \subseteq \pi$, there is at most one block, called a zero block, satisfying $B - B$.
- Edges are noncrossing when the integers $1, 2, \ldots, n, -1, -2, \ldots, -n$ are arranged in this order.

$\text{NC}^{(L)}(n) = \{ (\sigma, X) : \sigma \in \text{NC}(n), \ X \text{ is a set of nonnesting blocks of } \sigma \}$

Define $\#^b \sigma : \text{NC}_B(n) \rightarrow \text{NC}^{(L)}(n)$ as follows.

Proposition (2009, K.) : the first interpretation for $\text{NC}_B(n)$

The map $\#^b : \text{NC}_B(n) \rightarrow \text{NC}^{(L)}(n)$ is a bijection.

- $\#^b : \text{NC}_B(n) \rightarrow \text{NC}^{(L)}(n)$, $\#^b(\sigma)$ is the set of (\sigma, X) such that \sigma \in \text{NC}(n) and \ y \ is \ either \ the \ emptyset \ \emptyset \ or \ a \ block \ \sigma$.
- $\#^b(\sigma) = \text{NC}(n) \times [n+1]$.

For each Coxeter group, we can generalize Edelman’s method.

Proposition (2009, K.) : the second interpretation for $\text{NC}_B(n)$

$\eta^b = \#^b : \text{NC}_B(n) \rightarrow \text{NC}^{(L)}(n)$ is a bijection.

Ideas of Combinatorial Proof of Müller and Krattenthaler’s theorem

- For type A and type B, we can generalize Edelman’s method.
- For type D, we use the second interpretation, $\#^b(\sigma)$ for $\text{NC}_B(n)$.

Applications of the interpretations for $\text{NC}_B(n)$

The following gives a new bijection proof of $\#^b NC_B(n) = \frac{1}{n+1} \binom{2n}{n}$.

$\#^b NC_B(n) = \#^b NC^{(L)}(n) \implies \#^b NC_B(n) = [n+1]$.

- $\#^b NC^{(L)}(n)$ : the subset of $\#^b NC^{(L)}(n)$ whose elements are fixed under the 180° rotation in the circular representation.
- $\#^b NC^{(L)}(n) = \#^b NC^{(L)}(2n+1) \implies \#^b NC^{(L)}(2n+1) = \emptyset$.
- An element in $\#^b NC^{(L)}(n)$ for $\alpha = 4$ and $n = 7$.

Conjecture (2006, Armstrong)

The zeta polynomial (the number of multichains of length $i$) is equal to

$$Z(\text{NC}^{(L)}(2n+1), i) = \binom{2n+1}{n-i}.$$

- For $1 \leq i \leq n$, $\pi \in \text{NC}(2n+1)$ is augmented $i$-divisible if all but one blocks of $\pi$ have sizes divisible by $i$.
- $\text{NC}^{(L)}(2n+1)$ is the set of augmented $i$-divisible noncrossing partitions of $[2n+1]$.

Theorem (2007, Müller and Krattenthaler)

$\text{NC}^{(L)}(2n+1) \equiv \#^b NC^{(L)}(n) \#^b NC^{(L)}(n)$.

- As a corollary, we obtain Armstrong’s conjecture.

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