Chain enumeration of k-divisible noncrossing partitions of classical types

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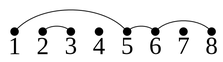
Introduction

Definition (1972, Kreweras) : Noncrossing Partitions

$\mathrm{NC}(n)$ is the set of **noncrossing partitions** of $[n]=\{1,2,\dots,n\}$.

- Each element B in a partition π is called a **block**.
- If the size of each block of $\pi\in\mathrm{NC}(n)$ is divisible by k, it is called a **k-divisible noncrossing partition**.
- For $\pi\in\mathrm{NC}(n)$, **block size vector** of π is $(b;b_1,b_2,\dots,b_n)$ where
  - b is the number of blocks of π,
  - b_i is the number of blocks of size i.
- The block size vector of the following is $(4;2,1,0,1,0,\dots,0)$:

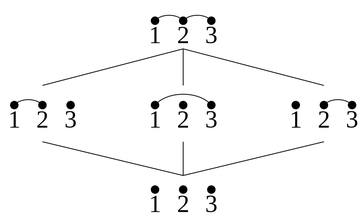


Theorem (1972, Kreweras)

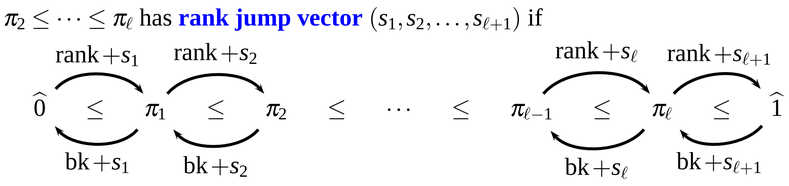
The number of $\pi\in\mathrm{NC}(n)$ with the **block size vector** $(b;b_1,b_2,\dots,b_n)$ is equal to

$$\frac{1}{b}\binom{b}{b_1,b_2,\dots,b_n}\binom{n}{b-1}.$$

- For $\pi_1,\pi_2\in\mathrm{NC}(n)$, we give the partial order $\pi_1\le\pi_2 \quad\Leftrightarrow\quad \pi_1$ is a **refinement** of π_2



- $\mathrm{NC}(n)$ is **graded** with $\mathrm{rank}(\pi)=n-\mathrm{bk}(\pi)$.
- A multichain $\pi_1\le\pi_2\le\cdots\le\pi_\ell$ has **rank jump vector** $(s_1,s_2,\dots,s_{\ell+1})$ if



Theorem (1980, Edelman)

The number of multichains $\pi_1\le\pi_2\le\cdots\le\pi_\ell$ in $\mathrm{NC}(n)$ with **rank jump vector** $(s_1,s_2,\dots,s_{\ell+1})$ is equal to

$$\frac{1}{n}\binom{n}{s_1}\binom{n}{s_2}\cdots\binom{n}{s_{\ell+1}}.$$

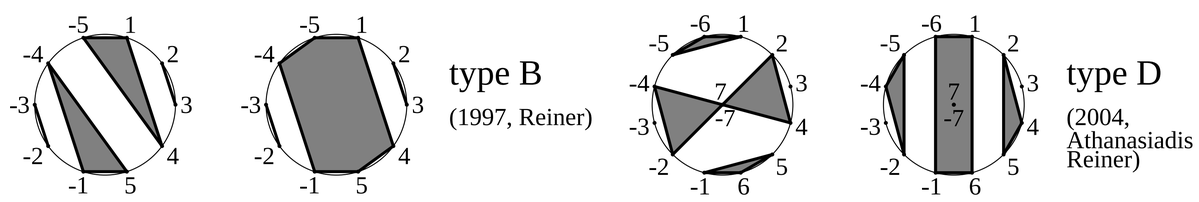
- The number of $\pi\in\mathrm{NC}(n)$ with k blocks is the **Narayana number**: $\frac{1}{n}\binom{n}{n-k}\binom{n}{k-1}=\frac{1}{n}\binom{n}{k}\binom{n}{k-1}$.
- The number of maximal chains in $\mathrm{NC}(n)$ is $\mathbf{n^{n-2}}$. (Stanley (1996) found a bijection between them with **parking functions**.)

Definition (2003, Bessis; Brady and Watt): Noncrossing partitions of a Coxeter group W

- (W,S) : a finite Coxeter system, $\quad T=\{wsw^{-1}:s\in S,w\in W\}$: the set of reflections
- Define $\ell_T(w):=\min\{k:w=t_1t_2\cdots t_k,t_i\in T\}$, and the partial order $u\le_T w \iff \ell_T(w)=\ell_T(u)+\ell_T(u^{-1}w)$
- For a Coxeter element c of W, for example $c=s_1s_2\cdots s_n$, define

$$\mathrm{NC}(W)\cong\mathrm{NC}(W,c)=\{w\in W:1\le_T w\le_T c\}.$$

Combinatorial models for classical Coxeter groups



- In 2006, Armstrong defined **k-divisible noncrossing partitions** $\mathrm{NC}^{(k)}(W)$ for each Coxeter group W.
- The combinatorial models $\mathrm{NC}^{(k)}(n)\cong\mathrm{NC}^{(k)}(A_{n-1})$ and $\mathrm{NC}^{(k)}_B(n)\cong\mathrm{NC}^{(k)}(B_n)$ are the natual ones.
- The model $\mathrm{NC}^{(k)}_D(n)\cong\mathrm{NC}^{(k)}(D_n)$ was found by Müller and Krattenthaler in 2007.

Theorem (2007, Müller and Krattenthaler)

The number of multichains $\pi_1\le\pi_2\le\cdots\le\pi_\ell$ in $\mathrm{NC}^{(k)}(n)$ with **rank jump vector** $(s_1,s_2,\dots,s_{\ell+1})$ such that the **block size vector** of π_1 is $(b;b_1,b_2,\dots,b_n)$ is $\frac{1}{b}\binom{b}{b_1,b_2,\dots,b_n}\binom{kn}{s_2}\cdots\binom{kn}{s_{\ell+1}}$.

The number of such multichains in $\mathrm{NC}^{(k)}_B(n)$ is $\binom{b}{b_1,b_2,\dots,b_n}\binom{kn}{s_2}\cdots\binom{kn}{s_{\ell+1}}$.

The number of such multichains in $\mathrm{NC}^{(k)}_D(n)$ is $\binom{b}{b_1,b_2,\dots,b_n}\binom{k(n-1)}{s_2}\cdots\binom{k(n-1)}{s_{\ell+1}}$, if $b_1+2b_2+\cdots+nb_n\le n-2$,

$$2\binom{b}{b_1,b_2,\dots,b_n}\binom{k(n-1)}{s_2}\cdots\binom{k(n-1)}{s_{\ell+1}}+\frac{s_i-1}{b-1}\binom{b-1}{b_1-1,b_2,\dots,b_n}\sum_{i=2}^{\ell+1}\binom{k(n-1)}{s_2}\cdots\binom{k(n-1)}{s_i-1}\cdots\binom{k(n-1)}{s_{\ell+1}},$$

if $b_1+2b_2+\cdots+nb_n=n$.

Main purpose : Prove the above theorem combinatorially!

- Müller and Krattenthaler mentioned that finding a combinatorial proof for type D seems rather hopeless.
- Since type D noncrossing partitions are special cases of type B noncrossing partitions, understanding type B noncrossing partitions is important.
- We will find a connection between $\mathrm{NC}_B(n)$ and the easiest object $\mathrm{NC}(n)$.

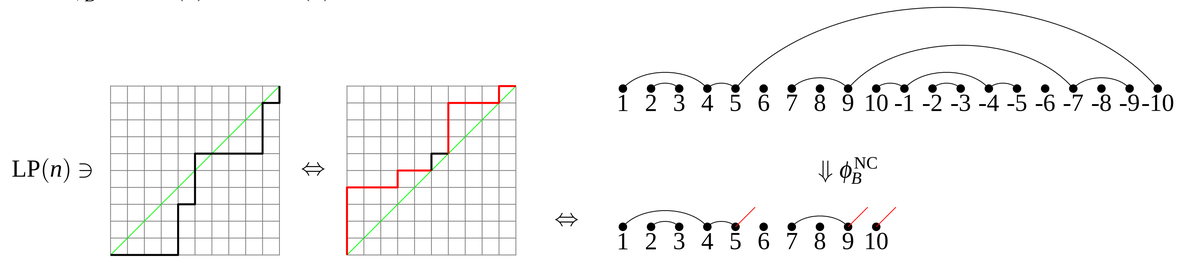
Two interpretations for $\mathrm{NC}_B(n)$

Equivalent definition of $\mathrm{NC}_B(n)$

$\mathrm{NC}_B(n)$ is the set of partitions π of $[\pm n]=\{1,2,\dots,n,-1,-2,\dots,-n\}$ such that

- if $B\in\pi$ then $-B\in\pi$,
- there is at most one block, called **zero block**, satisfying $B=-B$,
- edges are noncrossing when the integers $1,2,\dots,n,-1,-2,\dots,-n$ are arranged in this order.

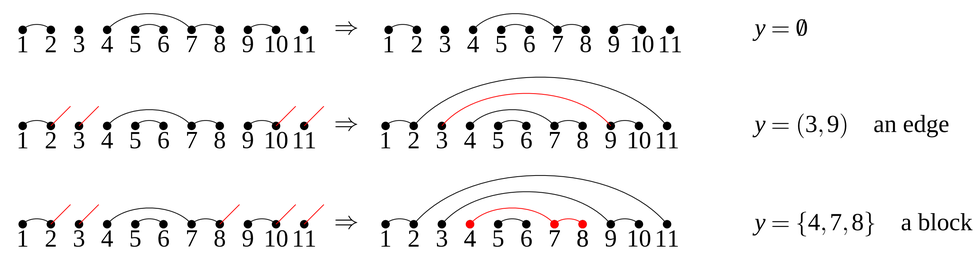
- $\mathrm{NC}^{\mathrm{NN}}(n)=\{(\sigma,X):\sigma\in\mathrm{NC}(n),\ X$ is a set of **nonnested blocks** of $\sigma\}$
- Define $\phi^{\mathrm{NC}}_B:\mathrm{NC}_B(n)\to\mathrm{NC}^{\mathrm{NN}}(n)$ as follows.



Proposition (2009, K.) : the first interpretation for $\mathrm{NC}_B(n)$

The map $\phi^{\mathrm{NC}}_B:\mathrm{NC}_B(n)\to\mathrm{NC}^{\mathrm{NN}}(n)$ is a bijection.

- $\mathscr{B}(n)$: the set of (σ,y) such that $\sigma\in\mathrm{NC}(n)$ and y is either **the emptyset** $\emptyset$, an **edge** of σ, or a **block** of σ.
- Note that $\mathscr{B}(n)=\mathrm{NC}(n)\times[n+1]$.
- For $(\sigma,X)\in\mathrm{NC}^{\mathrm{NN}}(n)$, define $\varphi_B(\sigma,X)$ as follows:



Proposition (2009, K.): the second interpretation for $\mathrm{NC}_B(n)$

$\psi_B=\varphi_B\circ\phi^{\mathrm{NC}}_B:\mathrm{NC}_B(n)\to\mathscr{B}(n)$ is a bijection.

Ideas of Combinatorial Proof of Müller and Krattenthaler's theorem

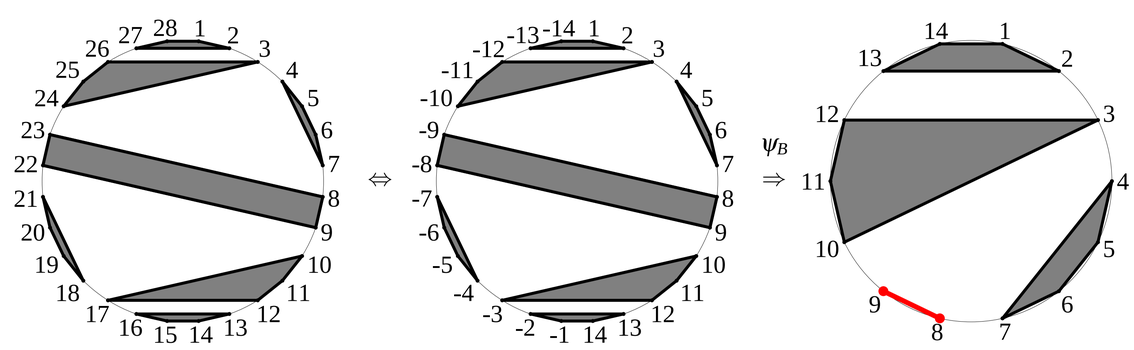
- For type A and type B, we can generalize Edelman's method.
- For type D, use the second interpretation $\mathscr{B}(n)$ for $\mathrm{NC}_B(n)$.

Applications of the interpretations for $\mathrm{NC}_B(n)$

The following gives **a new bijective proof** of $\#\mathrm{NC}(n)=\frac{1}{n+1}\binom{2n}{n}$:

$$\begin{array}{ccccc}\mathrm{NC}_B(n) & — & \mathrm{NC}^{\mathrm{NN}}(n) & — & \mathscr{B}(n)=\mathrm{NC}(n)\times[n+1]\\ & & | & & \\ & & \mathrm{LP}(n) & & \end{array}$$

- $\widetilde{\mathrm{NC}}^{(k)}(n)$: the subposet of $\mathrm{NC}^{(k)}(n)$ whose elements are **fixed under the 180° rotation** in the circular representation.
- $\widetilde{\mathrm{NC}}^{(k)}(2m)\cong\mathrm{NC}^{(k)}_B(m)$, $\quad\widetilde{\mathrm{NC}}^{(2d+1)}(2m+1)=\emptyset$, $\quad$ **the interesting case** $\widetilde{\mathrm{NC}}^{(2d)}(2m+1)=?$
- An element in $\widetilde{\mathrm{NC}}^{(k)}(n)$ for $k=4$ and $n=7$.



Conjecture (2006, Armstrong)

The **zeta polynomial** (the number of multichains of length ℓ) is equal to:

$$Z(\widetilde{\mathrm{NC}}^{(2k)}(2n+1),\ell)=\binom{n+\ell(2kn+k)}{n}.$$

- For $1\le r<k$, $\pi\in\mathrm{NC}(kn+r)$ is **augmented k-divisible** if all but one blocks of π have sizes divisible by k.
- $\mathrm{NC}^{(2k)}(n;r)$: the poset of augmented k-divisible noncrossing partitions of $[kn+r]$

Theorem (2009, K.)

$$\widetilde{\mathrm{NC}}^{(2k)}(2n+1)\cong\mathrm{NC}^{(2k)}(n;k).$$

- As a corollary, we obtain Armstrong's conjecture.

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