# Combinatorics of the PASEP partition function

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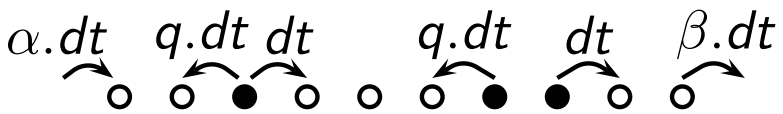
# Introduction

The Partially Asymmetric Self-Exclusion Process (PASEP) is a probabilistic model describing the evolution of particles in a finite number $N$ of sites.




In the time interval $dt$, possible events are:

- if $\circ$, the leftmost site becomes $\bullet$ with probability $\alpha.dt$
- if $\bullet$, the rightmost site becomes $\circ$ with probability $\beta.dt$
- $\bullet\circ$ becomes $\circ\bullet$ with probability $dt$
- $\circ\bullet$ becomes $\bullet\circ$ with probability $q.dt$

# Introduction

The partition function $Z_N$ is the normalization constant for stationary probabilities (such that non-normalized probabilities of each state is *polynomial* in the parameters). $Z_N$ is a polynomial in $\frac{1}{\alpha}$, $\frac{1}{\beta}$, $q$.

Many physical quantities can be obtained from the partition function (phase diagram, correlation functions, currents, density profile...)

There are various ways to obtain the partition function:

- Matrix-product form of $Z_N$ [Derrida & al]
- Generating function of lattice paths [Brak & al]
- Generating function of permutation tableaux, permutations [Corteel Williams]

# Introduction

## Theorem (Derrida & al)

*Let $D$ and $E$ be matrices, $W$ a row vector, $V$ a column vector, such that:*

$$DE - qED = D + E, \qquad WE = \frac{1}{\alpha}W, \qquad DV = \frac{1}{\beta}V, \qquad WV = 1,$$

*then for any $m$ word in $D$ and $E$ of length $N$,*

- *$WmV$ defines a unique polynomial in $\frac{1}{\alpha}$, $\frac{1}{\beta}$, $q$,*
- *and it is the non-normalized probability of the state $m$ (under $D \leftrightarrow \bullet$, $E \leftrightarrow \circ$)*

## Example

$DED = qEDD + DD + ED$, so the non-normalized probability of $\bullet \circ \bullet$ is

$$WDEDV = \frac{q}{\alpha\beta^2} + \frac{1}{\beta^2} + \frac{1}{\alpha\beta}.$$

It follows that the sum of non-normalized probabilities, is

$$Z_N = \sum_{m \in \{D,E\}^N} WmV = W(D+E)^N V$$

It is interesting to consider an extra variable $y$ and define

$$Z_N(y) = W(yD+E)^N V$$

This way, the coefficient of $y^k$ in $Z_N(y)$ corresponds to states with exactly $k$ particles.

Example

$$\begin{aligned} Z_2(y) &= W\Big[y^2 D^2 + y(DE+ED) + EE\Big] V \\ &= W\Big[y^2 D^2 + y\big((1+q)ED + D + E\big) + EE\Big] V \\ &= \frac{y^2}{\beta^2} + y\Big(\frac{1+q}{\alpha\beta} + \frac{1}{\beta} + \frac{1}{\alpha}\Big) + \frac{1}{\alpha^2} \end{aligned}$$

Let $\tilde{\alpha} = (1-q)\frac{1}{\alpha} - 1$, $\tilde{\beta} = (1-q)\frac{1}{\beta} - 1$.
It is possible to give explicit $D$, $E$, $W$, $V$ satisfying
$DE - qED = D + E$, $WE = \frac{1}{\alpha}W$, $DV = \frac{1}{\beta}V$, $WV = 1$ [Derrida & al].

$$D = \frac{1}{1-q}\begin{pmatrix} 1+\tilde{\beta} & 1-q & & & (0) \\ & 1+\tilde{\beta}q & 1-q^2 & & \\ & & 1+\tilde{\beta}q^2 & 1-q^3 & \\ & & & 1+\tilde{\beta}q^3 & \ddots \\ (0) & & & & \ddots \end{pmatrix},$$

$$E = \frac{1}{1-q}\begin{pmatrix} 1+\tilde{\alpha} & & & & (0) \\ 1-\tilde{\alpha}\tilde{\beta} & 1+\tilde{\alpha}q & & & \\ & 1-\tilde{\alpha}\tilde{\beta}q & 1+\tilde{\alpha}q^2 & & \\ & & 1-\tilde{\alpha}\tilde{\beta}q^2 & 1+\tilde{\alpha}q^3 & \\ (0) & & & \ddots & \ddots \end{pmatrix},$$

$$W = (1,0,0,\dots), \qquad V = (1,0,0,\dots)^*.$$

# Introduction

$yD + E$ is tridiagonal and can be thought as a transfer matrix, then

$$Z_N(y) = (1, 0, 0, \dots)(yD + E)^N \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

shows that $Z_N(y)$ is a generating function of weighted Motzkin paths.

# Introduction

Motivated by the PASEP and the links with permutations, we want to evaluate $Z_N$, characterized by

$$(1-q)^N Z_N = \sum_{\substack{\text{Motzkin path } P \\ \text{of length } N}} w(P)$$

where the weight $w(P)$ is the product of

- $1 - q^{h+1}$ for a step $\nearrow$ at height $h$ to $h+1$,
- $(1+y) + (\tilde{\alpha} + y\tilde{\beta})q^h$ for a step $\rightarrow$ at height $h$,
- $y(1 - \tilde{\alpha}\tilde{\beta}q^{h-1})$ for a step $\searrow$ at height $h$ to $h-1$,

where $\tilde{\alpha} = (1-q)\frac{1}{\alpha} - 1$, $\tilde{\beta} = (1-q)\frac{1}{\beta} - 1$.

# Introduction

## Theorem (Blythe & al)

$$Z_N(1) = \frac{1}{(1-q)^N} \sum_{n=0}^{N} R_{N,n} B_n$$

*where*

$$R_{N,n} = \sum_{i=0}^{\lfloor \frac{N-n}{2} \rfloor} (-1)^i q^{\binom{i+1}{2}} \begin{bmatrix} n+i \\ i \end{bmatrix}_q \left( \binom{2N}{N-n-2i} - \binom{2N}{N-n-2i-2} \right)$$

*and*

$$B_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \tilde{\alpha}^k \tilde{\beta}^{n-k}.$$

## Proof.

Diagonalize $D + E$ in terms of $q$-Hermite polynomials, this gives an integral which can be simplified analytically. $\square$

# Introduction

## Theorem

$$Z_N(y) = \frac{1}{(1-q)^N} \sum_{n=0}^{N} R_{N,n} B_n$$

*where*

$$R_{N,n} = \sum_{i=0}^{\lfloor \frac{N-n}{2} \rfloor} (-y)^i q^{\binom{i+1}{2}} \begin{bmatrix} n+i \\ i \end{bmatrix}_q \sum_{j=0}^{N-n-2i} y^j \left( \binom{N}{j} \binom{N}{n+2i+j} - \binom{N}{j-1} \binom{N}{n+2i+j+1} \right)$$

*and*

$$B_n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \tilde{\alpha}^k (y\tilde{\beta})^{n-k}.$$

This will be proved bijectively.

# Outline

To prove $(1-q)^N Z_N = \sum_{n=0}^{N} R_{N,n} B_n$, the scheme is the following.

- Step 0. Define some sets $\mathfrak{Z}_N$, $\mathfrak{R}_{N,n}$, $\mathfrak{B}_n$ of weighted Motzkin paths, $\mathfrak{Z}_N$ having generating function $(1-q)^N Z_N$
- Step 1. Give a weight-preserving bijection $\mathfrak{Z}_N \to \bigsqcup_{n=0}^{N} \mathfrak{R}_{N,n} \times \mathfrak{B}_n$
- Step 2. Show that $\mathfrak{R}_{N,n}$ has generating function $R_{N,n}$
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# Step 0

$(1-q)^N Z_N$ is the g.f. of Motzkin paths of length $N$, with weights:

- $1 - q^{h+1}$ for a step $\nearrow$ at height $h$ to $h+1$,
- $(1+y) + (\tilde{\alpha} + y\tilde{\beta})q^h$ for a step $\rightarrow$ at height $h$,
- $y(1 - \tilde{\alpha}\tilde{\beta}q^{h-1})$ for a step $\searrow$ at height $h$ to $h-1$.

Instead of a weight $1 - q^{h+1}$ on steps $\nearrow$, we can write

$$1 - q^{h+1} = (1-q) + (q - q^2) + \cdots + (q^h - q^{h+1})$$

and consider that the weight is $q^i - q^{i+1}$ with $0 \leq i \leq h$ for a step $\nearrow$ at height $h$ to $h+1$.

# Step 0

Let $\mathfrak{Z}_N$ be the set of Motzkin paths of length $N$ with weights

- ▷ $q^i - q^{i+1}$ with $0 \leq i \leq h$ for a step $\nearrow$ at height $h$ to $h+1$,
- ▷ $1 + y$ or $(\tilde{\alpha} + y\tilde{\beta})q^h$ for a step $\rightarrow$ at height $h$
- ▷ $y$ or $-y\tilde{\alpha}\tilde{\beta}q^{h-1}$ for a step $\searrow$ at height $h$ to $h-1$.

It is bigger than the original set of Motzkin paths since there are several possible choices on each step but the generating function is also $(1-q)^N Z_N$.

# Step 0

Let $\mathfrak{R}_{N,n}$ be the set of Motzkin paths of length $N$ with weights

- $q^i - q^{i+1}$ with $0 \le i \le h$ for a step $\nearrow$ at height $h$ to $h+1$,
- $1 + y$ or $q^h$ for a step $\rightarrow$ at height $h$
  $n$ steps having a weight $q^h$,
- $y$ for a step $\searrow$

Let $\mathfrak{B}_n$ be the set of Motzkin paths of length $n$ with weights

- $q^i - q^{i+1}$ with $0 \le i \le h$ for a step $\nearrow$ at height $h$ to $h+1$,
- $(\tilde{\alpha} + y\tilde{\beta})q^h$ for a step $\rightarrow$ at height $h$,
- $-y\tilde{\alpha}\tilde{\beta}q^{h-1}$ for a step $\searrow$ at height $h$ to $h-1$.

# Outline

To prove $(1-q)^N Z_N = \sum_{n=0}^{N} R_{N,n} B_n$, the scheme is the following.

- Step 0. Define some sets $\mathfrak{Z}_N$, $\mathfrak{R}_{N,n}$, $\mathfrak{B}_n$ of weighted Motzkin paths, $\mathfrak{Z}_N$ having generating function $(1-q)^N Z_N$
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# Step 1

The bijection is best described in the direction

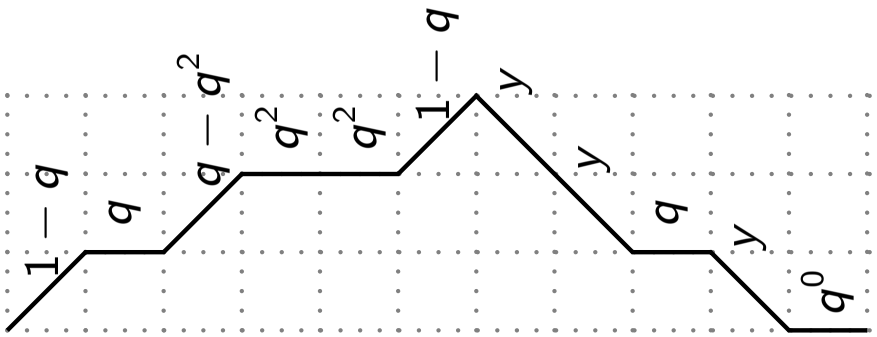
$$\bigsqcup_{n=0}^{N} \mathfrak{R}_{N,n} \times \mathfrak{B}_n \longrightarrow \mathfrak{Z}_N$$

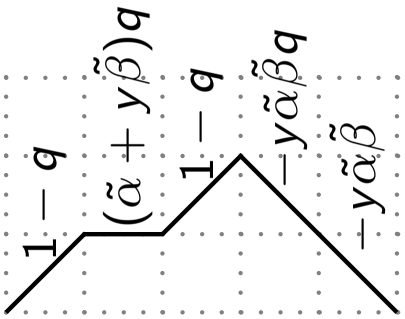
Let $(H_1, H_2) \in \mathfrak{R}_{N,n} \times \mathfrak{B}_n$. We build $\Lambda(H_1, H_2) \in \mathfrak{Z}_N$.

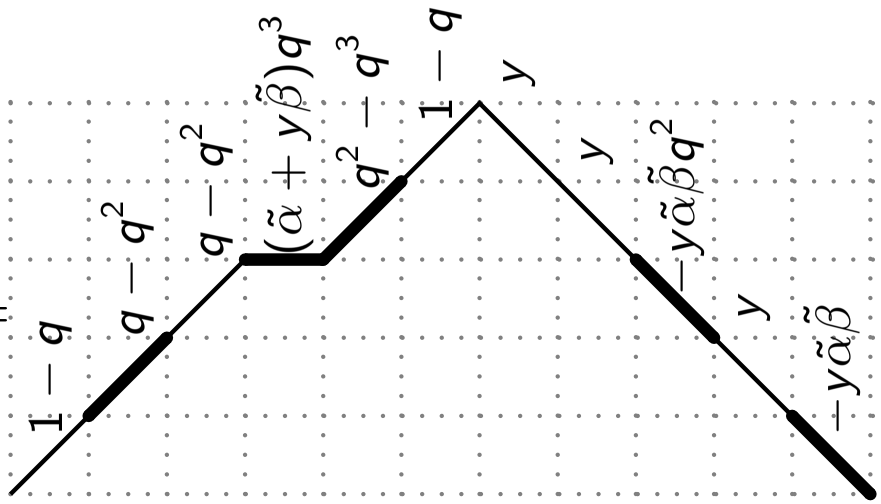
The definition is given via an example.

# Step 1

Let $(H_1, H_2) \in \mathfrak{R}_{N,n} \times \mathfrak{B}_n$. We build $\Lambda(H_1, H_2) \in \mathfrak{Z}_N$:

$H_1 =$ 


$H_2 =$ 


$\Lambda(H_1, H_2) =$ 

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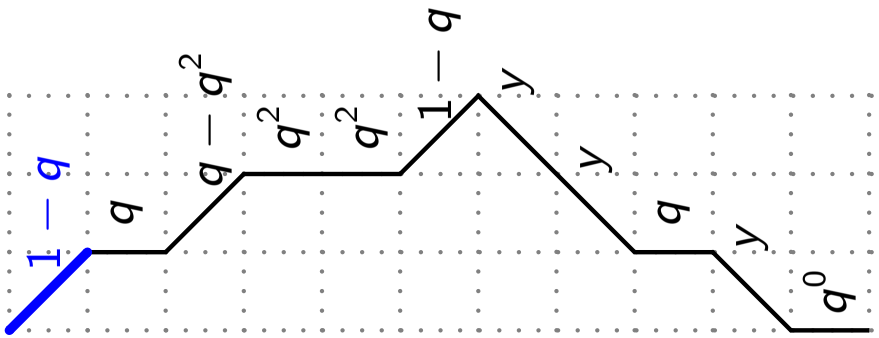
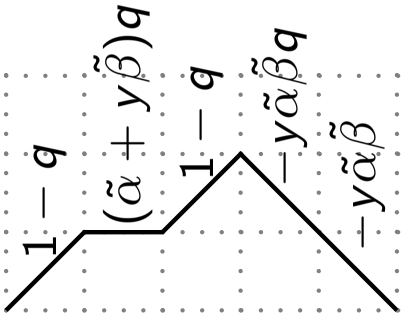
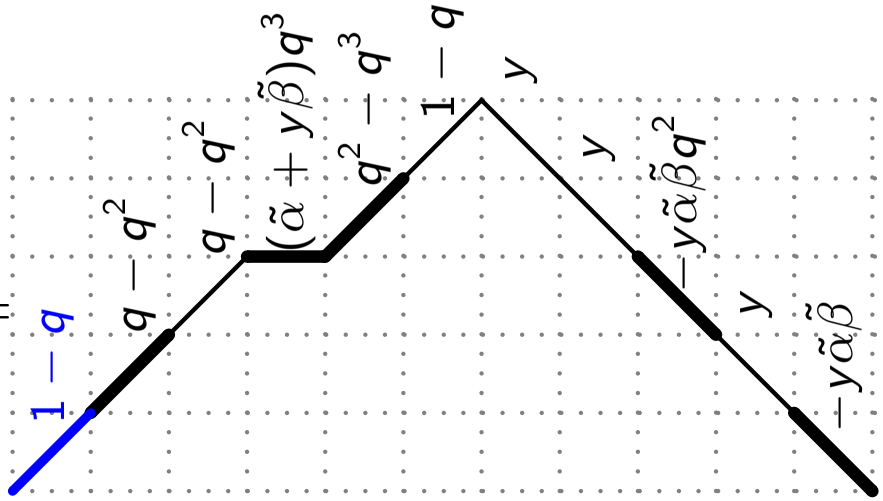
$H_1 =$ 


$H_2 =$ 


$\Lambda(H_1, H_2) =$ 

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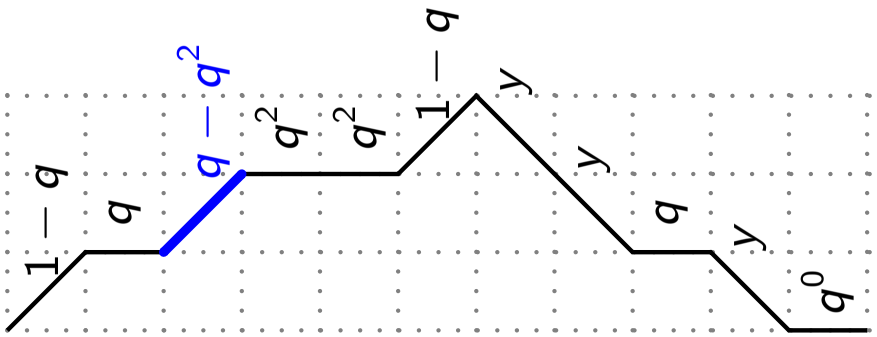
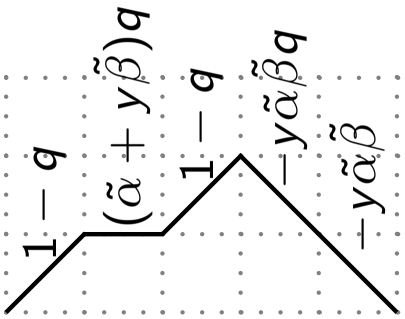
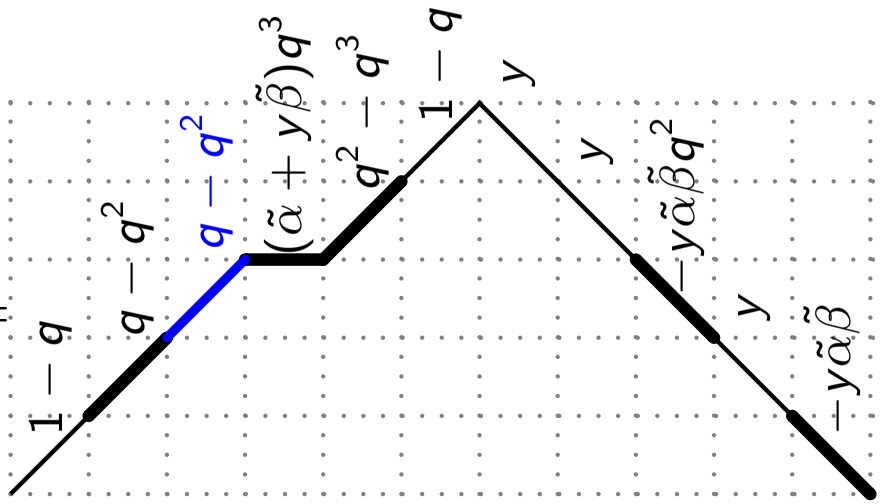
$H_1 =$ 


$H_2 =$ 


$\Lambda(H_1, H_2) =$ 

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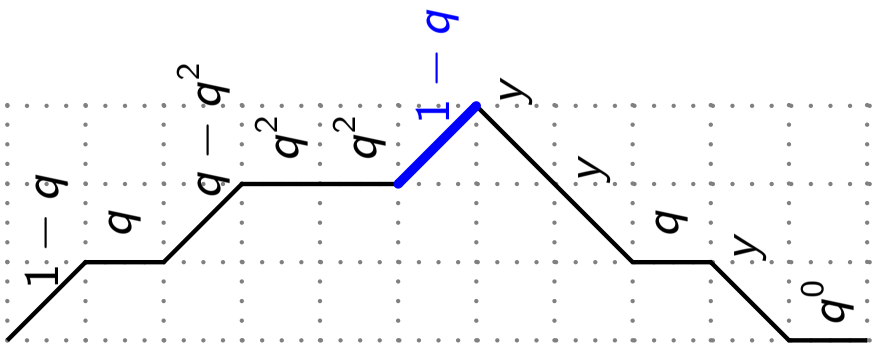
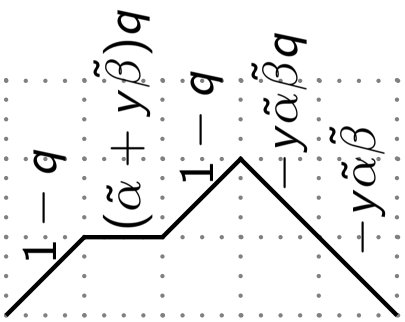
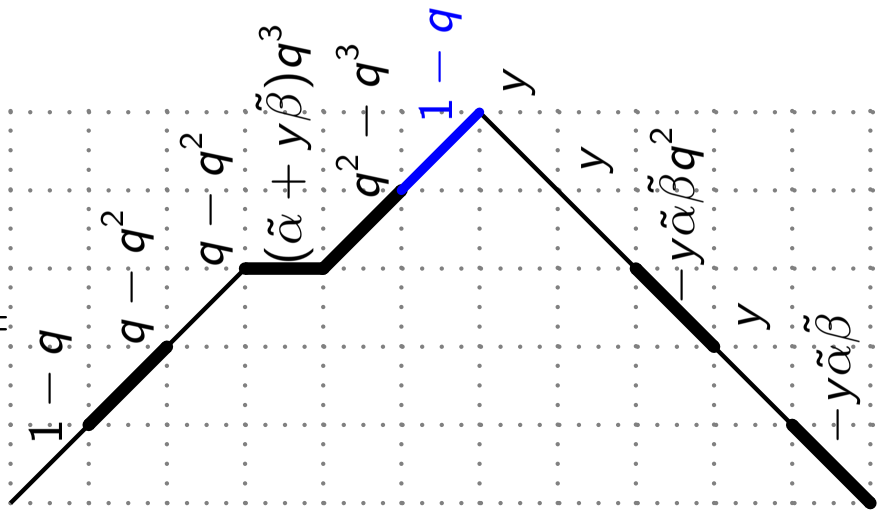
$H_1 =$ 


$H_2 =$ 


$\Lambda(H_1, H_2) =$ 

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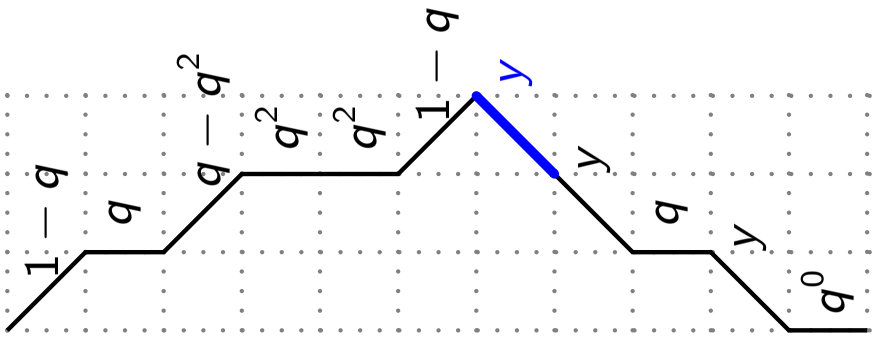
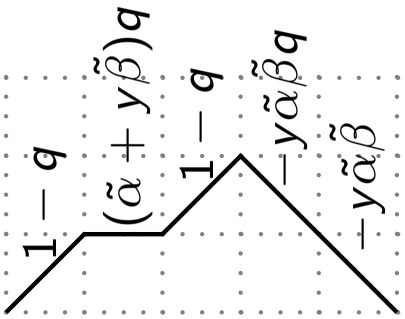
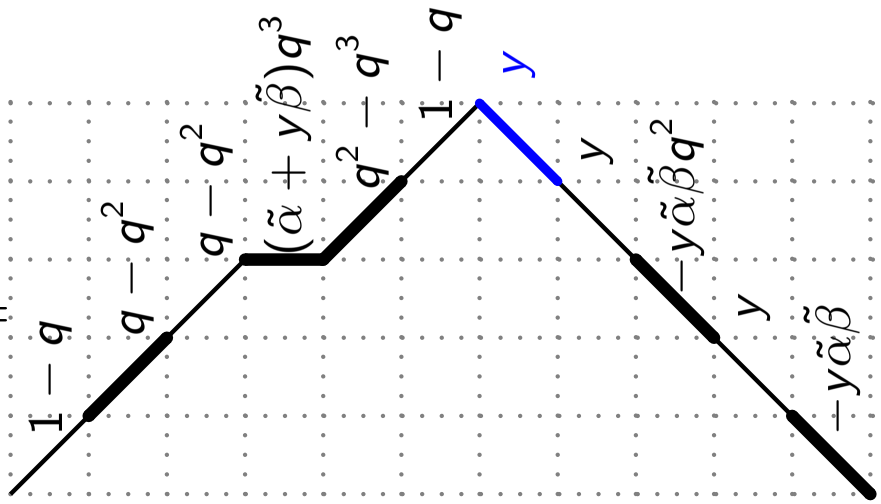
$H_1 =$ 


$H_2 =$ 


$\Lambda(H_1, H_2) =$ 

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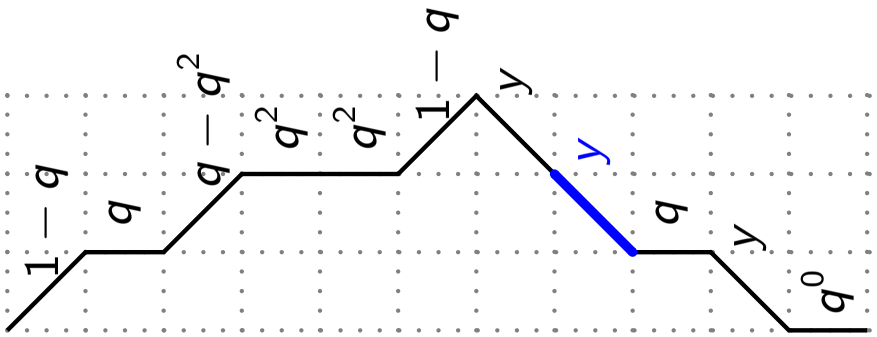
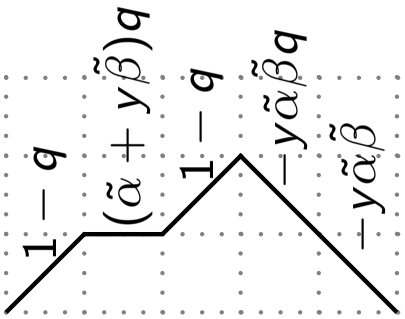
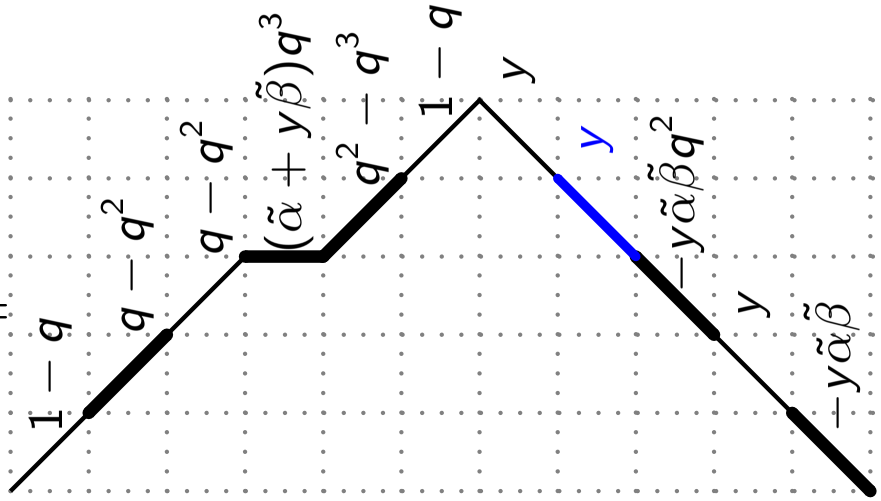
$H_1 =$ 


$H_2 =$ 


$\Lambda(H_1, H_2) =$ 

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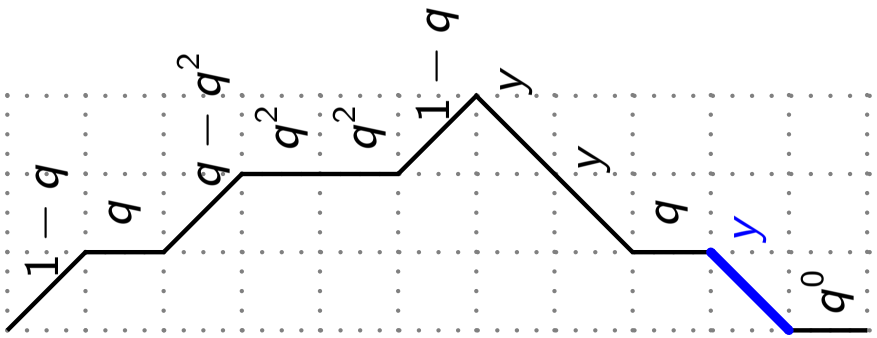
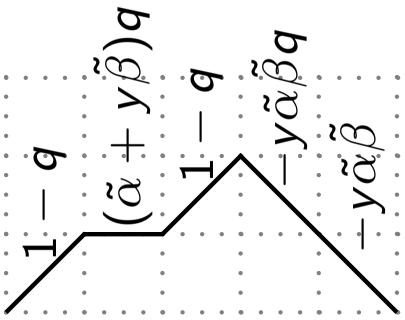
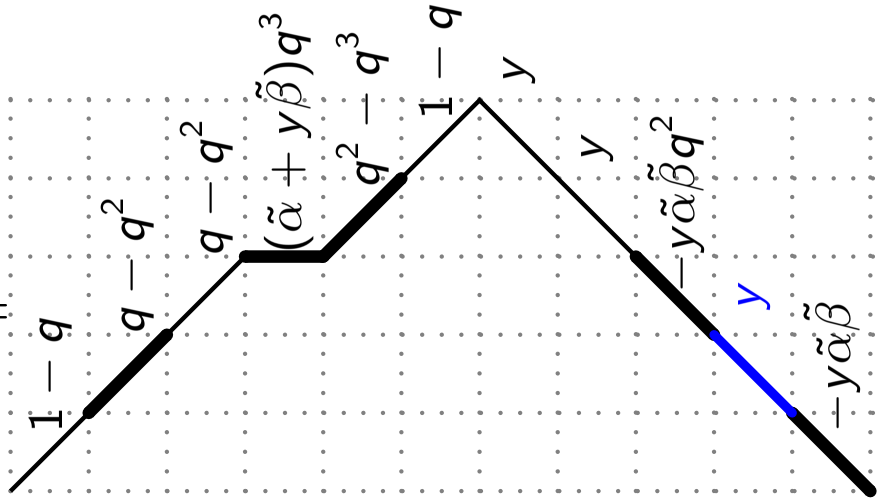
$H_1 =$ 


$H_2 =$ 


$\Lambda(H_1, H_2) =$ 

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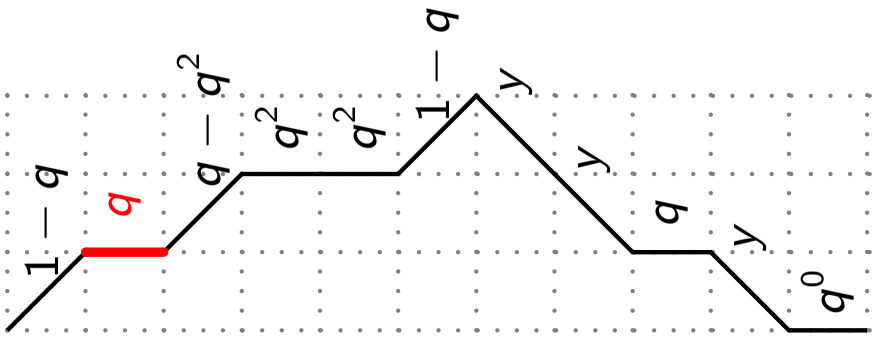
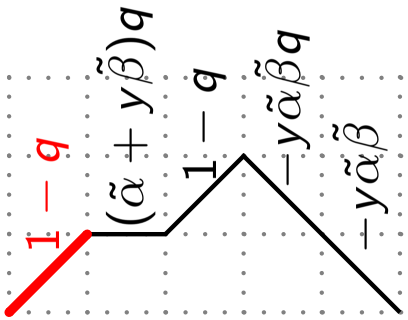
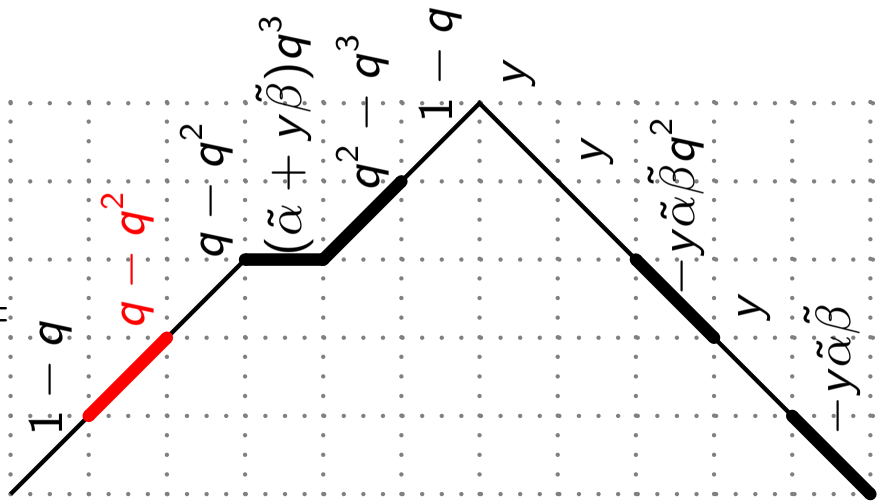
$H_1 =$ 


$H_2 =$ 


$\Lambda(H_1, H_2) =$ 

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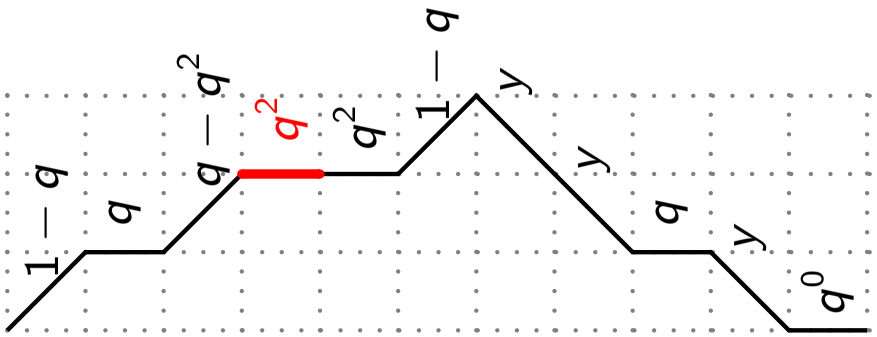
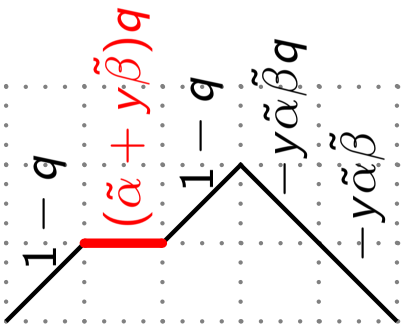
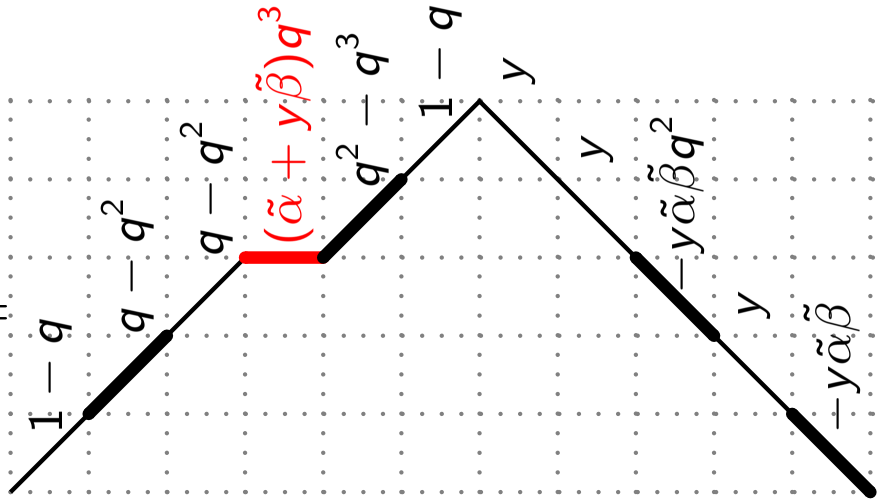
$H_1 =$ 


$H_2 =$ 


$\Lambda(H_1, H_2) =$ 

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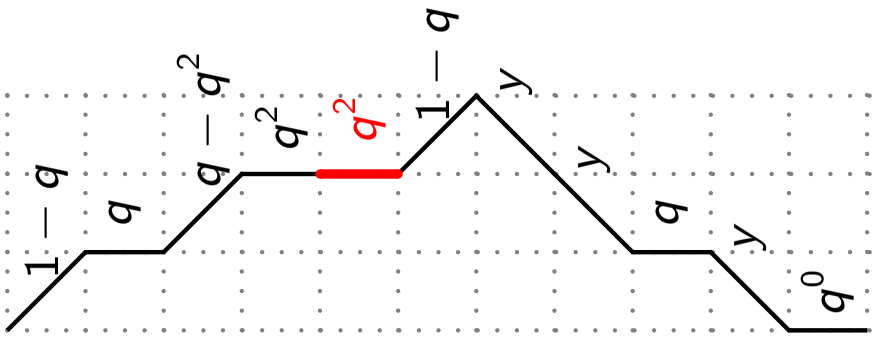
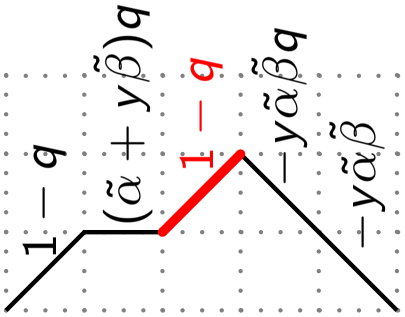
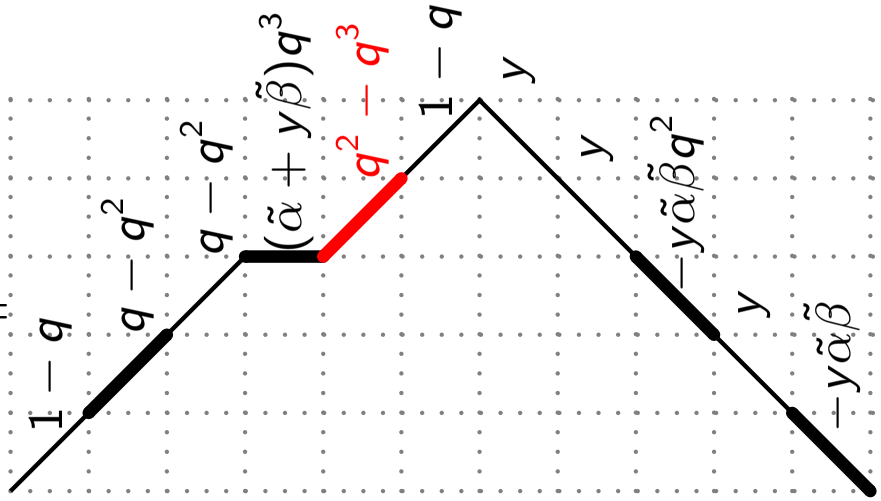
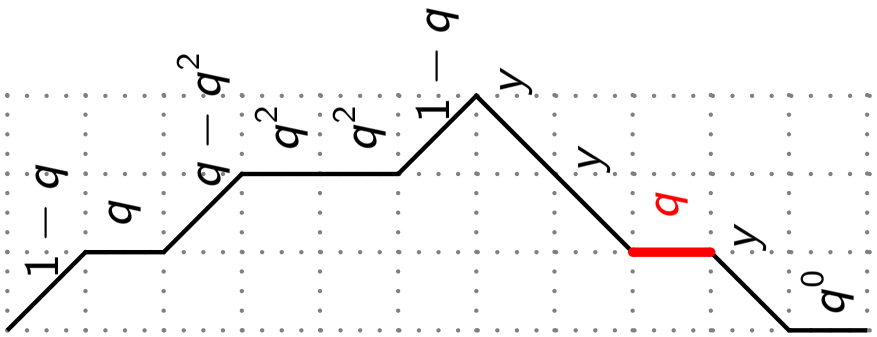
$H_1 =$ 

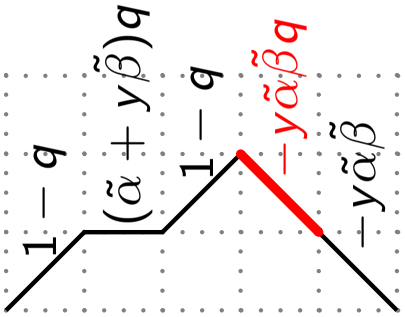

$H_2 =$ 

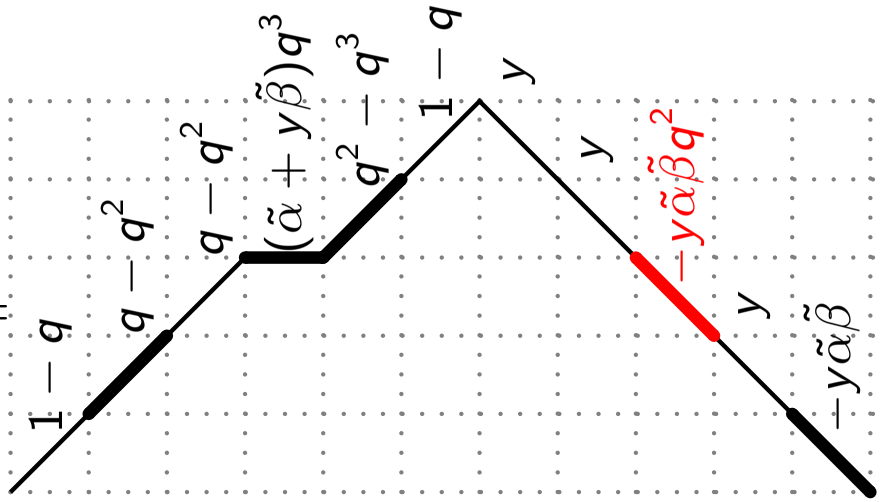

$\Lambda(H_1, H_2) =$ 

# Step 1

Let $(H_1, H_2) \in \mathfrak{R}_{N,n} \times \mathfrak{B}_n$. We build $\Lambda(H_1, H_2) \in \mathfrak{Z}_N$:

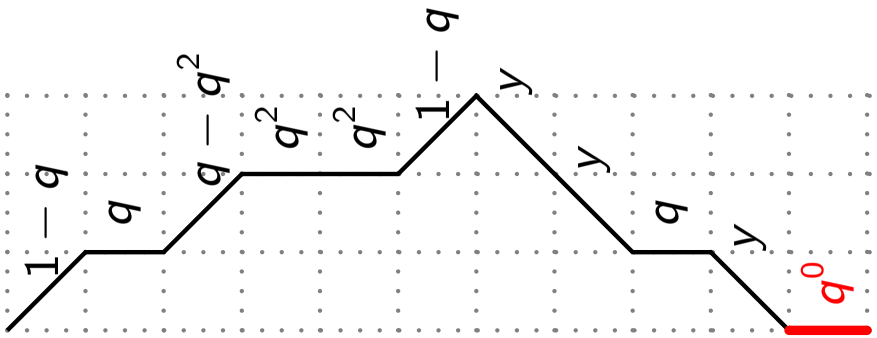
$H_1 =$ 

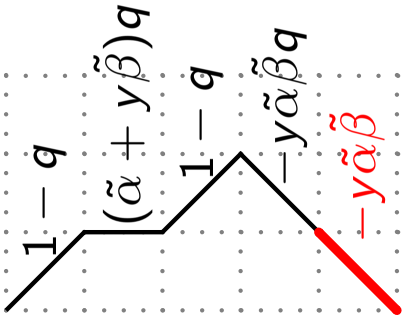

$H_2 =$ 

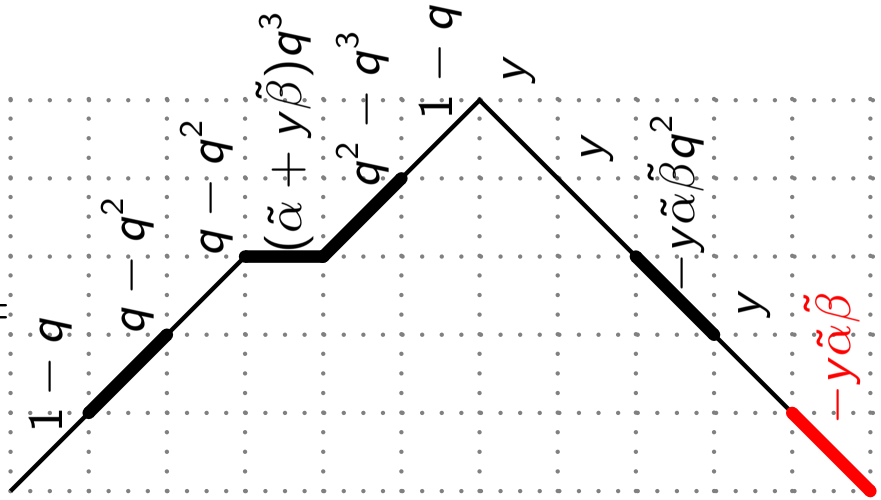

$\Lambda(H_1, H_2) =$ 

# Step 1

Let $(H_1, H_2) \in \mathfrak{R}_{N,n} \times \mathfrak{B}_n$. We build $\Lambda(H_1, H_2) \in \mathfrak{Z}_N$:

$H_1 =$ 


$H_2 =$ 


$\Lambda(H_1, H_2) =$ 

# Step 1

One can read $\Lambda(H_1, H_2)$ from right to left and recover the paths $H_1$, $H_2$, this proves injectivity.

If we forget the weights, there is a simple bijection between the two sets $\mathfrak{Z}_N$ and $\bigsqcup_{n=0}^{N} \mathfrak{R}_{N,n} \times \mathfrak{B}_n$. (these are essentially bicolored involutions), this shows equality of cardinal.

Thus $\Lambda$ is a weight-preserving bijection

$$\Lambda : \bigsqcup_{n=0}^{N} \mathfrak{R}_{N,n} \times \mathfrak{B}_n \to \mathfrak{Z}_N$$

# Outline

To prove $(1-q)^N Z_N = \sum_{n=0}^{N} R_{N,n} B_n$, the scheme is the following.

- Step 0. Define some sets $\mathfrak{Z}_N$, $\mathfrak{R}_{N,n}$, $\mathfrak{B}_n$ of weighted Motzkin paths, $\mathfrak{Z}_N$ having generating function $(1-q)^N Z_N$
- Step 1. Give a weight-preserving bijection $\mathfrak{Z}_N \to \bigsqcup_{n=0}^{N} \mathfrak{R}_{N,n} \times \mathfrak{B}_n$
- Step 2. Show that $\mathfrak{R}_{N,n}$ has generating function $R_{N,n}$
- Step 3. Show that $\mathfrak{B}_n$ has generating function $B_n$

# Step 2-3 (sketch)

Step 2: follows from the two different proofs for the particular case $Z_N(\alpha = \beta = 1)$ [Corteel, JV, Rubey, Prellberg]

Step 3: follows from known results on Al-Salam-Carlitz orthogonal polynomials.

# Outline

To prove $(1-q)^N Z_N = \sum_{n=0}^{N} R_{N,n} B_n$, the scheme is the following.

- Step 0. Define some sets $\mathfrak{Z}_N$, $\mathfrak{R}_{N,n}$, $\mathfrak{B}_n$ of weighted Motzkin paths, $\mathfrak{Z}_N$ having generating function $(1-q)^N Z_N$
- Step 1. Give a weight-preserving bijection $\mathfrak{Z}_N \to \bigsqcup_{n=0}^{N} \mathfrak{R}_{N,n} \times \mathfrak{B}_n$
- Step 2. Show that $\mathfrak{R}_{N,n}$ has generating function $R_{N,n}$
- Step 3. Show that $\mathfrak{B}_n$ has generating function $B_n$

$\square$

# Conclusion

We have a second proof of the formula for $Z_N(y)$ based on the operator point of view, using $\hat{D} = (q-1)D + I$ and $\hat{E} = (q-1)E + I$ which satisfy $\hat{D}\hat{E} - q\hat{E}\hat{D} = 1 - q$, and combinatorics of rook placements in Young diagrams.

$Z_n$ is essentially the moments of (rescaled) Al-Salam-Chihara orthogonal polynomials. There are analytical derivations of these moments and some more general moments [Ismail Stanton].

thanks
for your
attention