Combinatorics of the PASEP partition function

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The Partially Asymmetric Self-Exclusion Process (PASEP) is a probabilistic model describing the evolution of particles in a finite number $N$ of sites.

\[
\begin{array}{ccccccc}
\alpha dt & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ \\
\beta dt & \circ & \bullet & \circ & \bullet & \circ & \bullet & \circ \\
\end{array}
\]

In the time interval $dt$, possible events are:

- if $\circ$, the leftmost site becomes $\bullet$ with probability $\alpha dt$
- if $\bullet$, the rightmost site becomes $\circ$ with probability $\beta dt$
- $\bullet\circ$ becomes $\circ\bullet$ with probability $dt$
- $\circ\bullet$ becomes $\bullet\circ$ with probability $q dt$
The partition function $Z_N$ is the normalization constant for stationary probabilities (such that non-normalized probabilities of each state is \textit{polynomial} in the parameters). $Z_N$ is a polynomial in $\frac{1}{\alpha}, \frac{1}{\beta}, q$.

Many physical quantities can be obtained from the partition function (phase diagram, correlation functions, currents, density profile...)

There are various ways to obtain the partition function:

- Matrix-product form of $Z_N$ [Derrida & al]
- Generating function of lattice paths [Brak & al]
- Generating function of permutation tableaux, permutations [Corteel Williams]
Theorem (Derrida & al)

Let $D$ and $E$ be matrices, $W$ a row vector, $V$ a column vector, such that:

$$DE - qED = D + E,$$

$$WE = \frac{1}{\alpha}W,$$

$$DV = \frac{1}{\beta}V,$$

$$WV = 1,$$

then for any $m$ word in $D$ and $E$ of length $N$,

- $WmV$ defines a unique polynomial in $\frac{1}{\alpha}$, $\frac{1}{\beta}$, $q$,
- and it is the non-normalized probability of the state $m$ (under $D \leftrightarrow \bullet$, $E \leftrightarrow \circ$)

Example

$$DED = qEDD + DD + ED,$$ so the non-normalized probability of

$\bullet \circ \bullet$ is

$$WDEDV = \frac{q}{\alpha \beta^2} + \frac{1}{\beta^2} + \frac{1}{\alpha \beta}.$$
It follows that the sum of non-normalized probabilities, is

\[ Z_N = \sum_{m \in \{D, E\}^N} WmV = W(D + E)^N V \]

It is interesting to consider an extra variable \( y \) and define

\[ Z_N(y) = W(yD + E)^N V \]

This way, the coefficient of \( y^k \) in \( Z_N(y) \) corresponds to states with exactly \( k \) particles.

Example

\[ Z_2(y) = W \left[ y^2 D^2 + y(DE + ED) + EE \right] V \]

\[ = W \left[ y^2 D^2 + y \left( (1 + q)ED + D + E \right) + EE \right] V \]

\[ = \frac{y^2}{\beta^2} + y \left( \frac{1 + q}{\alpha \beta} + \frac{1}{\beta} + \frac{1}{\alpha} \right) + \frac{1}{\alpha^2} \]
Let \( \tilde{\alpha} = (1 - q)^{\frac{1}{\alpha}} - 1 \), \( \tilde{\beta} = (1 - q)^{\frac{1}{\beta}} - 1 \). It is possible to give explicit \( D, E, W, V \) satisfying
\[ DE - qED = D + E, \quad WE = \frac{1}{\alpha} W, \quad DV = \frac{1}{\beta} V, \quad WV = 1 \] [Derrida & al].

\[
D = \frac{1}{1 - q} \begin{pmatrix}
1 + \tilde{\beta} & 1 - q & (0) \\
1 + \tilde{\beta}q & 1 - q^2 & (0) \\
1 + \tilde{\beta}q^2 & 1 - q^3 & (0)
\end{pmatrix},
\]

\[
E = \frac{1}{1 - q} \begin{pmatrix}
1 + \tilde{\alpha} & 1 - q & (0) \\
1 + \tilde{\alpha}q & 1 + \tilde{\alpha}q^2 & (0) \\
1 + \tilde{\alpha}q^2 & 1 + \tilde{\alpha}q^3 & (0)
\end{pmatrix},
\]

\[ W = (1, 0, 0, \ldots), \quad V = (1, 0, 0, \ldots)^*. \]
Introduction

$yD + E$ is tridiagonal and can be thought as a transfer matrix, then

$$Z_N(y) = (1, 0, 0, \ldots) (yD + E)^N \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

shows that $Z_N(y)$ is a generating function of weighted Motzkin paths.
Introduction

Motivated by the PASEP and the links with permutations, we want to evaluate $Z_N$, characterized by

$$(1 - q)^N Z_N = \sum_{\text{Motzkin path } P} w(P)$$

where the weight $w(P)$ is the product of

- $1 - q^{h+1}$ for a step $\uparrow$ at height $h$ to $h + 1$,
- $(1 + y) + (\tilde{\alpha} + y\tilde{\beta})q^h$ for a step $\rightarrow$ at height $h$,
- $y(1 - \tilde{\alpha}\tilde{\beta}q^{h-1})$ for a step $\searrow$ at height $h$ to $h - 1$,

where $\tilde{\alpha} = (1 - q)\frac{1}{\alpha} - 1$, $\tilde{\beta} = (1 - q)\frac{1}{\beta} - 1$. 
Introduction

Theorem (Blythe & al)

\[ Z_N(1) = \frac{1}{(1 - q)^N} \sum_{n=0}^{N} R_{N,n} B_n \]

where

\[ R_{N,n} = \sum_{i=0}^\left[ \frac{N-n}{2} \right] (-1)^i q^{\left(\frac{i+1}{2}\right)} \binom{n+i}{i}_q \left( \binom{2N}{n-n-2i} - \binom{2N}{n-n-2i-2} \right) \]

and

\[ B_n = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q \tilde{\alpha}^k \tilde{\beta}^{n-k} \]

Proof.
Diagonalize \( D + E \) in terms of \( q \)-Hermite polynomials, this gives an integral which can be simplified analytically.
Introduction

**Theorem**

\[
Z_N(y) = \frac{1}{(1 - q)^N} \sum_{n=0}^{N} R_{N,n} B_n
\]

where

\[
R_{N,n} = \sum_{i=0}^{\left\lfloor \frac{N-n}{2} \right\rfloor} (-y)^i q^{(i+1)\left\lceil \frac{n+i}{2} \right\rceil} \sum_{j=0}^{N-n-2i} y^j \left( \binom{N}{j} \binom{N}{n+2i+j} - \binom{N}{j-1} \binom{N}{n+2i+j+1} \right)
\]

and

\[
B_n = \sum_{k=0}^{n} \left\lceil \begin{array}{c} n \\ k \end{array} \right\rceil \tilde{\alpha}^k (y \tilde{\beta})^{n-k}.
\]

This will be proved bijectively.
To prove \((1 - q)^N Z_N = \sum_{n=0}^{N} R_{N,n} B_n\), the scheme is the following.

- **Step 0.** Define some sets \(Z_N, R_{N,n}, B_n\) of weighted Motzkin paths, \(Z_N\) having generating function \((1 - q)^N Z_N\).
- **Step 1.** Give a weight-preserving bijection \(Z_N \rightarrow \bigcup_{n=0}^{N} R_{N,n} \times B_n\).
- **Step 2.** Show that \(R_{N,n}\) has generating function \(R_{N,n}\).
- **Step 3.** Show that \(B_n\) has generating function \(B_n\).
To prove \((1 - q)^N Z_N = \sum_{n=0}^{N} R_{N,n} B_n\), the scheme is the following.

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- **Step 3.** Show that \(B_n\) has generating function \(B_n\).
\[(1 - q)^N Z_N\] is the g.f. of Motzkin paths of length \(N\), with weights:

- \(1 - q^{h+1}\) for a step \(\Rightarrow\) at height \(h\) to \(h + 1\),
- \((1 + y) + (\tilde{\alpha} + y\tilde{\beta})q^h\) for a step \(\rightarrow\) at height \(h\),
- \(y(1 - \tilde{\alpha}\tilde{\beta}q^{h-1})\) for a step \(\Downarrow\) at height \(h\) to \(h - 1\).

Instead of a weight \(1 - q^{h+1}\) on steps \(\Rightarrow\), we can write

\[1 - q^{h+1} = (1 - q) + (q - q^2) + \cdots + (q^h - q^{h+1})\]

and consider that the weight is \(q^i - q^{i+1}\) with \(0 \leq i \leq h\) for a step \(\Rightarrow\) at height \(h\) to \(h + 1\).
Let $Z_N$ be the set of Motzkin paths of length $N$ with weights

\[ q^i - q^{i+1} \text{ with } 0 \leq i \leq h \] for a step $\nearrow$ at height $h$ to $h + 1$,

\[ 1 + y \text{ or } (\tilde{\alpha} + y\tilde{\beta})q^h \] for a step $\rightarrow$ at height $h$

\[ y \text{ or } -y\tilde{\alpha}\tilde{\beta}q^{h-1} \] for a step $\searrow$ at height $h$ to $h - 1$.

It is bigger than the original set of Motzkin paths since there are several possible choices on each step but the generating function is also $(1 - q)^N Z_N$. 
Step 0

Let $\mathcal{M}_{N,n}$ be the set of Motzkin paths of length $N$ with weights

- $q^i - q^{i+1}$ with $0 \leq i \leq h$ for a step $\nearrow$ at height $h$ to $h + 1$,
- $1 + y$ or $q^h$ for a step $\rightarrow$ at height $h$, $n$ steps having a weight $q^h$,
- $y$ for a step $\searrow$

Let $\mathcal{B}_n$ be the set of Motzkin paths of length $n$ with weights

- $q^i - q^{i+1}$ with $0 \leq i \leq h$ for a step $\nearrow$ at height $h$ to $h + 1$,
- $(\tilde{\alpha} + y \tilde{\beta}) q^h$ for a step $\rightarrow$ at height $h$,
- $-y \tilde{\alpha} \tilde{\beta} q^{h-1}$ for a step $\searrow$ at height $h$ to $h - 1$. 
To prove \((1 - q)^N Z_N = \sum_{n=0}^{N} R_{N,n} B_n\), the scheme is the following.

- **Step 0.** Define some sets \(Z_N, R_{N,n}, B_n\) of weighted Motzkin paths, \(Z_N\) having generating function \((1 - q)^N Z_N\)
- **Step 1.** Give a weight-preserving bijection \(Z_N \rightarrow \bigsqcup_{n=0}^{N} R_{N,n} \times B_n\)
- **Step 2.** Show that \(R_{N,n}\) has generating function \(R_{N,n}\)
- **Step 3.** Show that \(B_n\) has generating function \(B_n\)
Step 1

The bijection is best described in the direction

$$\bigcup_{n=0}^{N} \mathcal{K}_{N,n} \times \mathcal{B}_n \rightarrow \mathcal{Z}_N$$

Let \((H_1, H_2) \in \mathcal{K}_{N,n} \times \mathcal{B}_n\). We build \(\Lambda(H_1, H_2) \in \mathcal{Z}_N\).

The definition is given via an example.
Let \((H_1, H_2) \in \mathbb{R}^{N 	imes n}\). We build \(\Lambda(H_1, H_2) \in \mathbb{Z}^N\):

\[
\begin{align*}
H_1 &= 1 - q (\tilde{\alpha} + \tilde{\beta} y) - q y (\tilde{\alpha} \tilde{\beta} q^2 - y \tilde{\alpha} \tilde{\beta} q) = H_1 \quad \text{for } b - 1 < y < b - 2 \\
H_2 &= 1 - q (\tilde{\alpha} + \tilde{\beta} y) - q y (\tilde{\alpha} \tilde{\beta} q^2 - y \tilde{\alpha} \tilde{\beta} q) = H_2 \quad \text{for } b - 1 < y < b - 2
\end{align*}
\]

\[
\Lambda(H_1, H_2) = \frac{18}{35}
\]
Let $(H_1, H_2) \in \mathbb{R}^n \times \mathbb{R}^n$. We build $N \in \mathcal{F}_n \times \mathcal{F}_n$.

Step 1

$$\Lambda(H_1, H_2) = \mathcal{H}$$
Let \((H_1, H_2) \in \mathcal{N}_n \times \mathcal{B}_n\). We build \(\Lambda(H_1, H_2) \in \mathcal{N}_n\):
Let \((H_1, H_2) \in \mathcal{H}_{N, n} \times \mathcal{B}_{N, n}\). We build \(\Lambda(H_1, H_2) \in \mathbb{Z}_N\):
Let $(\mathcal{H}_1, \mathcal{H}_2) \in \mathbb{R}^N$, $n \times B_n$. We build $\Lambda(\mathcal{H}_1, \mathcal{H}_2) \in \mathbb{Z}^N$:

$$\begin{align*}
\mathcal{H}_1 &= 1 - q (\tilde{\alpha} + y \tilde{\beta}) \\
\mathcal{H}_2 &= 1 - q (\tilde{\alpha} + y \tilde{\beta})
\end{align*}$$

\(\Lambda(\mathcal{H}_1, \mathcal{H}_2) = \frac{22}{35}\)
Let $(H_1, H_2) \in \mathbb{R}^{n, B_n}$. We build $\Lambda(H_1, H_2) \in \mathbb{Z}^{n}$:
Step 1

Let \((H_1, H_2) \in \mathbb{R}^N_{n \times b_n}\). We build \(\Lambda(H_1, H_2) \in \mathbb{Z}^N\):

\[
\begin{align*}
H_1 & = 1 - q \tilde{\alpha} + y \tilde{\beta} \\
H_2 & = 1 - q \tilde{\alpha} + y \tilde{\beta}
\end{align*}
\]

\(\Lambda(H_1, H_2) = \left(\mathcal{H}_1, \mathcal{H}_2\right)_N\)

\[
\begin{align*}
\mathcal{H}_1 & = \mathcal{H} \\
\mathcal{H}_2 & = \mathcal{H}
\end{align*}
\]
Let \((H_1, H_2) \in \mathcal{R}_N \times \mathcal{B}_n\). We build \(\Lambda(H_1, H_2) \in \mathcal{B}_n\):
Let \((H_1, H_2) \in \mathbb{R}^{N_N \times u} \times u \mathbb{N}\). We build \(V \in \mathcal{F} \). Let \((H_1, H_2) \in \mathbb{Z}^N \). Then \(H_1 \in \mathbb{Z}^N \).
Let $(\mathcal{H}_1, \mathcal{H}_2) \in \mathbb{R}^N \times \mathbb{R}^N$. We build $\mathcal{H}$.
Let \((\mathcal{H}_1, \mathcal{H}_2) \in \mathbb{R}^N_{\mathbb{N}}\times \mathbb{N}^u \times \mathbb{N}^u \times \mathcal{X}\). We build \(\mathcal{V}(\mathcal{H}_1, \mathcal{H}_2) \in \mathfrak{Z}\).
Step 1

Let \((H_1, H_2) \in \mathbb{R}^N_{n \times b_n}\). We build \(\Lambda(H_1, H_2) \in \mathbb{Z}^N\):

\[
1 - q \quad q \quad q \quad - q
\]

\[
H_1 = 1 - q (\tilde{\alpha} + y \tilde{\beta})
\]

\[
H_2 = 1 - q \quad q \quad - q 
\]

\[
(\tilde{\alpha} + y \tilde{\beta})
\]

\[
\Lambda(H_1, H_2) = \binom{29}{35}
\]
Step 1

One can read $\Lambda(H_1, H_2)$ from right to left and recover the paths $H_1, H_2$, this proves injectivity.

If we forget the weights, there is a simple bijection between the two sets $\mathcal{Z}_N$ and $\bigcup_{n=0}^N \mathcal{R}_{N,n} \times \mathcal{B}_n$. (these are essentially bicolored involutions), this shows equality of cardinal.

Thus $\Lambda$ is a weight-preserving bijection

$$\Lambda : \bigcup_{n=0}^N \mathcal{R}_{N,n} \times \mathcal{B}_n \rightarrow \mathcal{Z}_N$$
To prove \((1 - q)^N Z_N = \sum_{n=0}^{N} R_{N,n} B_n\), the scheme is the following.

- **Step 0.** Define some sets \(Z_N, R_{N,n}, B_n\) of weighted Motzkin paths, \(Z_N\) having generating function \((1 - q)^N Z_N\).

- **Step 1.** Give a weight-preserving bijection \(Z_N \rightarrow \bigcup_{n=0}^{N} R_{N,n} \times B_n\).

- **Step 2.** Show that \(R_{N,n}\) has generating function \(R_{N,n}\).

- **Step 3.** Show that \(B_n\) has generating function \(B_n\).
Step 2: follows from the two different proofs for the particular case $Z_N(\alpha = \beta = 1)$ [Corteel, JV, Rubey, Prellberg]

Step 3: follows from known results on Al-Salam-Carlitz orthogonal polynomials.
To prove \((1 - q)^N Z_N = \sum_{n=0}^{N} R_{N,n} B_n\), the scheme is the following.

- **Step 0.** Define some sets \(\mathcal{Z}_N, \mathcal{R}_{N,n}, \mathcal{B}_n\) of weighted Motzkin paths, \(\mathcal{Z}_N\) having generating function \((1 - q)^N Z_N\).

- **Step 1.** Give a weight-preserving bijection \(\mathcal{Z}_N \rightarrow \bigsqcup_{n=0}^{N} \mathcal{R}_{N,n} \times \mathcal{B}_n\).

- **Step 2.** Show that \(\mathcal{R}_{N,n}\) has generating function \(R_{N,n}\).

- **Step 3.** Show that \(\mathcal{B}_n\) has generating function \(B_n\).
Conclusion

We have a second proof of the formula for $Z_N(y)$ based on the operator point of view, using $\hat{D} = (q - 1)D + I$ and $\hat{E} = (q - 1)E + I$ which satisfy $\hat{D}\hat{E} - q\hat{E}\hat{D} = 1 - q$, and combinatorics of rook placements in Young diagrams.

$Z_n$ is essentially the moments of (rescaled) Al-Salam-Chihara orthogonal polynomials. There are analytical derivations of these moments and some more general moments [Ismail Stanton].
thanks
for your
attention