Affine structures and a tableau model for E_6 crystals

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Crystals of Quantum Groups

Notation:

- $\mathfrak{g}$ = semisimple Lie algebra
- I = index set for Dynkin diagram of $\mathfrak{g}$
- $\{\Lambda_i : i \in I\}$ = fundamental weights of $\mathfrak{g}$
- $\bigoplus \mathbb{Z}_{\geq 0}\Lambda_i$ = dominant integral weights, $P = \bigoplus_i \mathbb{Z}\Lambda_i$ = weight lattice
- $\{\alpha_i : i \in I\}$ = simple roots of $\mathfrak{g}$, $\{\alpha_i^\vee : i \in I\}$ = simple coroots of $\mathfrak{g}$
- $U_q(\mathfrak{g})$ = quantized universal enveloping algebra of $\mathfrak{g}$ (generated by e_i, f_i, q^h with relations)

A **(combinatorial) crystal** is a nonempty set B together with maps

$$\mathrm{wt} : B \to P \qquad e_i, f_i : B \to B \cup \{0\}$$

satisfying

$$f_i(b) = b' \iff e_i(b') = b \text{ for all } b, b' \in B$$
$$\mathrm{wt}(f_i(b)) = \mathrm{wt}(b) - \alpha_i \text{ whenever } f_i(b) \in B$$
$$\langle \alpha_i^\vee, \mathrm{wt}(b) \rangle = \varphi_i(b) - \varepsilon_i(b) \text{ for all } b \in B$$

where $\varepsilon_i(b) = \max\{n \geq 0 \mid e_i^n(b) \neq 0\}$ and $\varphi_i(b) = \max\{n \geq 0 \mid f_i^n(b) \neq 0\}$.
We visualize the crystal as an edge-labeled directed graph where $b \xrightarrow{i} b'$ whenever $f_i(b) = b'$.
The **tensor product rule** builds a new crystal $B \otimes B'$ on the set $B \times B'$.
There is a correspondence

irreducible integrable $U_q(\mathfrak{g})$-modules $\longleftrightarrow$ dominant integral weights $= \bigoplus \mathbb{Z}_{\geq 0}\Lambda_i$

Every such module $V(\lambda)$ has a **crystal basis** $B(\lambda)$ with unique "highest weight" element $b \in B$ such that $e_i(b) = 0$ for all $i \in I$. Moreover $\mathrm{wt}(b) = \lambda$.
By semisimplicity, every crystal is a union of connected components, each of which is a highest weight crystal: $B = \bigoplus B(\lambda)$.

Affine Crystals

- When $\mathfrak{g}$ is a Kac–Moody algebra of affine type, the $U_q(\mathfrak{g})$-modules are generally **infinite dimensional**, so we consider the subalgebra $U_q'(\mathfrak{g})$ (generated by $e_i, f_i, K_i^{\pm 1}$) instead.
- **Kirillov–Reshetikhin crystals** $B^{r,s}$ correspond to finite dimensional $U_q'(\mathfrak{g})$-modules (where $r \in I \setminus \{0\}$, $s \in \mathbb{Z}_{\geq 0}$).
- These can be used to build arbitrary highest-weight $U_q(\mathfrak{g})$-modules $V(\lambda)$ using the "Kyoto path model" that embeds $B(\lambda) \subset B^{r_1,s} \otimes B^{r_2,s} \otimes \cdots$ where λ has level s. Here, the **level** of λ is given by $\langle \lambda, c \rangle$ where c is the canonical central element.
- The Kirillov–Reshetikhin crystals restrict to classical crystals:

$$B^{r,s}|_{\text{edges with labels in } I \setminus \{0\}} \quad \text{is a } U_q(\overline{\mathfrak{g}})\text{-module}$$

where $\overline{\mathfrak{g}}$ = the finite subalgebra of $\mathfrak{g}$.

Generalized tableaux

Lenart and Postnikov (2008) have given an "alcove path model" for classical crystals in all types that we have implemented in Sage. Figure 1 shows the resulting fundamental crystal $B(\Lambda_1)$ in type E_6.

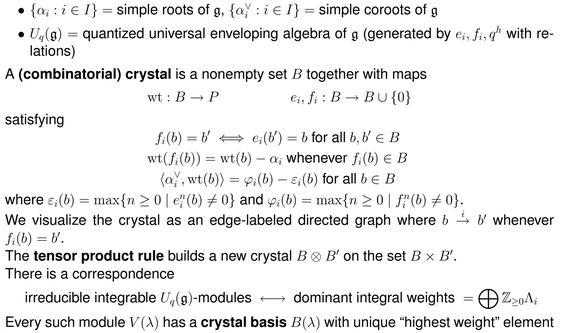


Figure 1: *Crystal graph for $B(\Lambda_1)$ of type E_6*

- $B(\Lambda_6)$ also has 27 nodes and is dual to $B(\Lambda_1)$ having all arrows reversed.
- We can unambiguously label any node of $B(\Lambda_1) \cup B(\Lambda_6)$ by weight.
- We compute the 0-weight at **level-zero** of $b \in B(\Lambda_1)$ from the classical weight $\mathrm{wt}(b)$ using the formula $0 = \langle \mathrm{wt}(b), c \rangle$ where c is the canonical central element.
- In E_6, every $B(\Lambda_i)$ for $i \in I \setminus \{0\}$ can be realized as a component of some tensor product of $B(\Lambda_1)$ and $B(\Lambda_6)$ factors.

$B(\Lambda_2)$	is generated by $21 \otimes 1$	in $B(\Lambda_6) \otimes B(\Lambda_1)$
$B(\Lambda_3)$	is generated by $13 \otimes 1$	in $B(\Lambda_1)^{\otimes 2}$
$B(\Lambda_4)$	is generated by $34 \otimes 13 \otimes 1$	in $B(\Lambda_1)^{\otimes 3}$
$B(\Lambda_5)$	is generated by $65 \otimes 6$	in $B(\Lambda_6)^{\otimes 2}$

- If b = unique highest weight node in $B(\lambda)$ and c = unique highest weight node in $B(\mu)$, then $B(\lambda + \mu)$ is generated by $b \otimes c \in B(\lambda) \otimes B(\mu)$.

Iterating this procedure gives a recursive description of any highest-weight crystal embedded in a tensor product of fundamental crystals. **Goal:** Give a *non-recursive* realization of such crystals.
(Cf. Kashiwara–Nakashima (1994) for tableaux in non-exceptional types.)

Definition 1. Let Λ_i and Λ_j be distinct fundamental weights, and $p, q \in \mathbb{Z}_{\geq 0}$. Then, we say that

$$b_1 \otimes b_2 \otimes \cdots \otimes b_k \in B(\Lambda_i)^{\otimes p} \otimes B(\Lambda_j)^{\otimes q}$$

is **weakly increasing** if

$$b_k \otimes b_{k+1} \in B(\mu_k + \mu_{k+1}) \subset B_k \otimes B_{k+1}$$

for each $1 \leq k < p + q$, where B_k is the kth tensor factor of $B(\Lambda_i)^{\otimes p} \otimes B(\Lambda_j)^{\otimes q}$ with highest weight $\mu_k \in \{\Lambda_i, \Lambda_j\}$.

Lemma 2. *The nodes of*

$$B(p\Lambda_i + q\Lambda_j) \subset B(\Lambda_i)^{\otimes p} \otimes B(\Lambda_j)^{\otimes q}$$

are precisely the **weakly increasing** *tensor products.*

Remark 3. It is not possible to obtain a pairwise weakly increasing condition that characterizes the nodes of $B(\Lambda_1 + \Lambda_6 + \Lambda_2)$. In the full paper, we give a more general result that explains how to get a "weakly increasing" condition that works for all highest weight crystals in any type.

For particular fundamental representations in E_6, we can be more specific about testing for the **weakly increasing** condition.

Proposition 4. *We have that $b_1 \otimes b_2 \in B(2\Lambda_1) \subset B(\Lambda_1)^{\otimes 2}$ if and only if there exists a directed path $b_1 \xrightarrow{i_1} \cdots \xrightarrow{i_p} b_2$ in $B(\Lambda_1)$.*

We can visualize the elements of $B(s\Lambda_1)$ as tableaux having a single row of s boxes filled with weakly increasing entries from $B(\Lambda_1)$. Using the tensor product rule, it is straightforward to read off the action of f_i on nodes represented in this way. For example,

$$\begin{array}{|c|c|c|c|c|}\hline \bar{0}\bar{1} & 2\bar{5} & 2\bar{5} & 4\bar{1}2 & 2\bar{1}0 \\ \hline \end{array} \in B(5\Lambda_1),$$

$$f_1\left(\begin{array}{|c|c|c|c|c|}\hline \bar{0}\bar{1} & 2\bar{5} & 2\bar{5} & 4\bar{1}2 & 2\bar{1}0 \\ \hline \end{array}\right) = \begin{array}{|c|c|c|c|c|}\hline \bar{0}\bar{1} & 2\bar{5} & 2\bar{5} & 4\bar{1}2 & \bar{1}\bar{2}30 \\ \hline \end{array}.$$

There are 78 nodes in $B(\Lambda_2)$, which is generated by

$$2\bar{1}\bar{0} \otimes \bar{0}1 \in B(\Lambda_6) \otimes B(\Lambda_1).$$

Proposition 5. *We have that*

$$(b_1 \otimes c_1) \otimes (b_2 \otimes c_2) \in B(2\Lambda_2) \subset (B(\Lambda_6) \otimes B(\Lambda_1))^{\otimes 2}$$

if and only if

(1) There exist directed paths $b_1 \xrightarrow{i_1} \cdots \xrightarrow{i_p} b_2$ in $B(\Lambda_6)$, and $c_1 \xrightarrow{i_1} \cdots \xrightarrow{i_q} c_2$ in $B(\Lambda_1)$, and

(2) If c_1 is dual to b_2, then there exists a directed path $(b_1 \otimes c_1) \xrightarrow{i_1} \cdots \xrightarrow{i_r} (b_2 \otimes c_2)$ of length ≥ 1 in $B(\Lambda_2)$.

For example, $(1\bar{3} \otimes \bar{1}3) \otimes (1\bar{3} \otimes \bar{1}3)$ is *not* weakly increasing. We can visualize the elements of $B(s\Lambda_2)$ as tableaux having two rows of s boxes:

$$\begin{array}{|c|c|c|c|c|}\hline 2\bar{1}\bar{0} & 4\bar{1}\bar{2} & 4\bar{1}\bar{2} & 0\bar{3}4 & 0\bar{1}3 \\ \hline \bar{0}\bar{1} & \bar{0}\bar{1} & \bar{0}\bar{1} & \bar{0}\bar{1}\bar{3} & \bar{0}\bar{3}4 \\ \hline \end{array} \in B(5\Lambda_2),$$

$$f_3\left(\begin{array}{|c|c|c|c|c|}\hline 2\bar{1}\bar{0} & 4\bar{1}\bar{2} & 4\bar{1}\bar{2} & 0\bar{3}4 & 0\bar{1}3 \\ \hline \bar{0}\bar{1} & \bar{0}\bar{1} & \bar{0}\bar{1} & \bar{0}\bar{1}\bar{3} & \bar{0}\bar{3}4 \\ \hline \end{array}\right) = \begin{array}{|c|c|c|c|c|}\hline 2\bar{1}\bar{0} & 4\bar{1}\bar{2} & 4\bar{1}\bar{2} & 0\bar{3}4 & 0\bar{1}3 \\ \hline \bar{0}\bar{1} & \bar{0}\bar{1} & \bar{0}\bar{1} & \bar{0}\bar{1}\bar{3} & \bar{0}\bar{3}4 \\ \hline \end{array}.$$

Affinization of crystals

Definition 6. Let $\dot{p}$ be an automorphism of the affine Dynkin diagram, and B be a classical crystal.
Say $\dot{p}$ **induces a twisted isomorphism of crystals** if there exists a bijection of crystals $p : B \cup \{0\} \to B' \cup \{0\}$ such that

$$p(b) = 0 \text{ if and only if } b = 0, \text{ and} \tag{1}$$
$$p \circ f_i(b) = f_{\dot{p}(i)} \circ p(b) \text{ and } p \circ e_i(b) = e_{\dot{p}(i)} \circ p(b) \tag{2}$$

for all $i \in I \setminus \{0, \dot{p}^{-1}(0)\}$ and all $b \in B$.

Definition 7. Let B be a directed graph with edge labels from I. Say B is **regular** if $B|_{\text{ edges with labels } i,j}$ is a classical crystal for all pairs $\{i, j\} \subset I$.

Definition 8. Let B be a classical crystal. Suppose $\widetilde{B}$ has the same nodes and edges as B together with some additional 0-edges. If $\widetilde{B}$ is regular, then we say that $\widetilde{B}$ is a **combinatorial affine structure** for B.

The Kirillov–Reshetikhin crystals that we study must satisfy this definition and have weights of level-zero by Kashiwara (2002, 2005).
Consequences:

- $\mathrm{wt}_i(b) = \mathrm{wt}_{\dot{p}(i)}(p(b))$ for all $b \in B$ and $i \in I \cup \{0\}$.
- We can define a combinatorial affine structure for B "by promotion" if we let

$$f_0 := p^{-1} f_{\dot{p}(0)} p.$$

For this to work, we must identify the image $p(B)$ with a canonically labeled classical crystal so that we can infer the $f_{\dot{p}(0)}$ edges.

Theorem 9. *Let B be a classical E_6 crystal. Suppose there exists a twisted isomorphism $p : B \to B$ induced by $\dot{p} : 0 \mapsto 1 \mapsto 6$ and p has order 3. Then, there exists a combinatorial affine structure on B.*

Proof sketch:

- Define 0-arrows on B by the map $p^{-1} f_1 p$. It suffices to show the resulting graph is regular: Each restriction $\widetilde{B}|_{0,i \text{ arrows}}$ must be an $A_1 \times A_1$ or A_2 crystal.
- Stembridge (2003) has given **local rules** depending only $\varphi_i(b)$ and $\varepsilon_i(b)$ that characterize the $A_1 \times A_1$ and A_2 crystals. Applying p preserves φ and ε, so this proves the restriction for $i = 1, 2, 3, 4, 5$. For $i = 6$, we use that p has order 3, and apply $p^{-1} = p^2$.

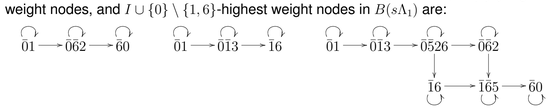
Composition graphs

- Suppose $J \subset I$, and let $H^J(B)$ denote the $(I \setminus J)$-highest weight nodes of B.
- Our general strategy to define a twisted isomorphism p on a classical crystal B is to first define p on $H^J(B)$, and then extend this definition to the rest of B using Equation (2).
- There exist **composition graphs** G_J such that every node of $H^J(B)$ can be realized as tensor product of vertices that form a directed path in G_J.

Theorem 10. *There exists a unique twisted isomorphism $p : B(s\Lambda_1) \to B(s\Lambda_1)$ of order 3.*

This gives the unique affine structure for the Kirillov–Reshetikhin crystal $B^{1,s} \cong B(s\Lambda_1)$ (and $B^{6,s}$ by duality).

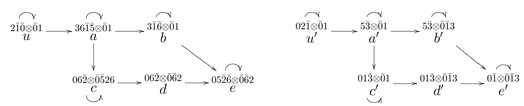
Proof: The composition graphs for the $I \setminus \{6\}$-highest weight nodes, $I \setminus \{1\}$-highest weight nodes, and $I \cup \{0\} \setminus \{1, 6\}$-highest weight nodes in $B(s\Lambda_1)$ are:



Hence, the $I \setminus \{6\}$-highest weight nodes are determined by their affine level-0 weight, and so their image under p is uniquely determined and it is straightforward to verify that it has order 3. □

Next, we consider the Kirillov–Reshetikhin crystal $B^{2,s}$ which has classical decomposition $\bigoplus_{j=0}^{s} B(j\Lambda_2)$ by work of Chari (2001), Kang–Kashiwara–Misra–Miwa–Nakashima–Nakayashiki (1992) and Kashiwara–Misra–Okado–Yamada (2007).

- The composition graphs for the $I \setminus \{6\}$-highest weight nodes and $I \setminus \{1\}$-highest weight nodes in $B(j\Lambda_2)$ are:



- Let $\mathcal{C}(m)$ denote the set

$$\{(L_2, L_3, L_5) \in \mathbb{Z}_{\geq 0} : L_2 + L_3 + L_5 = m\}$$

of **weak compositions** of m into 3 parts.

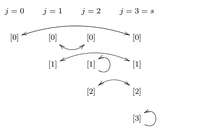
- There is a bijection from the $I \setminus \{6\}$-highest weight nodes of $B(j\Lambda_2)$ to $\bigcup_{m=0}^{j} \mathcal{C}(m)$ such that a node corresponding to the weak composition $L_2 + L_3 + L_5 = m$ has $I \setminus \{6\}$-weight $L_2\Lambda_2 + L_3\Lambda_3 + L_5\Lambda_5$.
- In particular, the $I \setminus \{6\}$-highest weight nodes of $B(j\Lambda_2)$ are determined by their $\{2, 3, 5\}$-weight, and moreover for any such node b, we have

$$j = \varphi_6(b) + \mathrm{wt}_2(b) + \mathrm{wt}_3(b) + \mathrm{wt}_5(b).$$

Theorem 11. *There is a unique twisted isomorphism*

$$p : \bigoplus_{j=0}^{s} B(j\Lambda_2) \to \bigoplus_{j=0}^{s} B(j\Lambda_2)$$

of order 3. This isomorphism sends $I \setminus \{6\}$-highest weight nodes to $I \setminus \{1\}$-highest weight nodes, twists the weights, and permutes components as illustrated for $s = 3$:

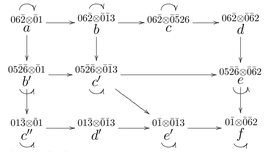


Here, $[m]$ denotes the set of nodes with $I \setminus \{6\}$-highest weight $L_2\Lambda_2 + L_3\Lambda_3 + L_5\Lambda_5$ satisfying $L_2 + L_3 + L_5 = m$.

Thus, we obtain a combinatorial affine structure on $\bigoplus_{j=0}^{s} B(j\Lambda_2)$, which is isomorphic to $B^{2,s}$ of type $E_6^{(1)}$.

Proof sketch:

- To prove that p has order 3, we need to consider the composition graph for the $I \cup \{0\} \setminus \{6, 1\}$-highest weight nodes:



Here, there are 6 maximal chains.

- Let b be an $I \setminus \{1\}$-highest weight node of $B(j\Lambda_2)$. Then, for every $j \leq k \leq s$, there exists a unique $I \setminus \{6, 1\}$-highest weight node b' in $B(k\Lambda_2)$ that is twisted isomorphic to b. Moreover, $\varphi_1(b') = k - j$.
- Hence, we consider

$$\begin{array}{c} I \setminus \{1\}, \text{ component } j \\ \downarrow p \\ I \setminus \{6, 1\}, \text{ component } k \end{array} \xrightarrow{p} \begin{array}{c} I \setminus \{6\}, \text{ component } j' \\ \text{encoded by } \varphi_6 \end{array}$$

Then,

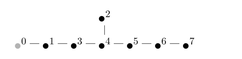
$$j' - (L_2 + L_3 + L_5) = \varphi_6(p^2(b)) = \varphi_1(p(b)) = (k - j) \geq 0.$$

We use an induction argument to show that $k = s$ always. □

Benkart, Frenkel, Kang and Lee (2006) gave a type-independent description for the perfect crystals of level 1 corresponding to the adjoint representation. Hernandez and Nakajima (2006) constructed the Kirillov–Reshetikhin crystals $B^{r,1}$ in types $E_6^{(1)}$ and $E_7^{(1)}$.

Further questions

We have a conjecture for the adjoint crystal $B^{1,s}$ in type $E_7^{(1)}$:



Conjecture 12. *Define $p : \bigoplus_{k=0}^{s} B(k\Lambda_1) \to \bigoplus_{k=0}^{s} B(k\Lambda_1)$ on the $I \setminus \{7\}$-highest weight nodes by sending $b \in B(k\Lambda_1)$ to the unique $I \setminus \{7\}$-highest weight node b' in component $(s - k) + (\mathrm{wt}_1(b) + \mathrm{wt}_2(b) + \mathrm{wt}_6(b))$ satisfying $\mathrm{wt}_{\dot{p}(i)}(b') = \mathrm{wt}_i(b)$ for each $i \in \{1, 2, 6\}$.*
Let $f_0 = p \circ f_7 \circ p$. Then f_0 commutes with f_7 so we obtain a combinatorial affine structure on $\bigoplus_{k=0}^{s} B(k\Lambda_1)$, which is isomorphic to $B^{1,s}$ of type $E_7^{(1)}$.