The Discrete Geometry of Moment Polytopes

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Simple polytopes

$\Delta \subseteq \mathbb{R}^d$
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$$\Delta \subseteq \mathbb{R}^d \rightsquigarrow \mathbf{f} = (f_{-1}, f_0, f_1, \ldots, f_{d-1})$$

$$f_k = \# \text{ k-dimensional faces}$$
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$\mathbf{f} = (1, 4, 4)$  $(1, 6, 6)$  $(1, 4, 6, 4)$  $(1, 8, 12, 6)$  $(1, 10, 15, 7)$
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\[ \rightsquigarrow \mathbf{h} = (h_0, h_1, \ldots, h_d) \]

\[ h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1} \]

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\[ f = (1, 4, 4), (1, 6, 6), (1, 4, 6, 4), (1, 8, 12, 6), (1, 10, 15, 7) \]

\[ h = (1, 2, 1), (1, 4, 1), (1, 1, 1, 1), (1, 3, 3, 1), (1, 4, 4, 1) \]

**Theorem** [Stanley, 1975]:

(a) \( h_k = h_{d-k} \)

(b) \( h_0 \leq h_1 \leq \ldots \leq h_{\left\lfloor \frac{d}{2} \right\rfloor} \)
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**Theorem** [Stanley, 1975]:

(a) \( h_k = h_{d-k} \) \( \iff \) Poincaré duality

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(a) \( h_k = h_{d-k} \) \(\leadsto\) Poincaré duality

(b) \( h_0 \leq h_1 \leq \cdots \leq h_{\lfloor \frac{d}{2} \rfloor} \) \(\leadsto\) hard Lefschetz property
Symplectic manifolds

A symplectic manifold is a manifold.
Symplectic manifolds

A symplectic manifold is a manifold with a two-form $\omega \in \Omega^2(M)$ that is:

- **Closed**: $d\omega = 0$
- **Non-degenerate**: $\omega^n = d\text{Vol} \iff M$ is $2n$-dimensional
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\((\mathbb{R}^2, \omega = dx \wedge dy) \sim (\mathbb{R}^{2n}, \omega = \sum dx_i \wedge dy_i)\)
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\[(\mathbb{R}^2, \omega = dx \wedge dy) \sim \sim \rightarrow (\mathbb{R}^{2n}, \omega = \sum dx_i \wedge dy_i)\]

**Darboux’s Theorem:**
We may always choose coordinates \(x_1, \ldots, x_n, y_1, \ldots, y_n\) on \(M\) so that locally

\[\omega = \sum dx_i \wedge dy_i.\]
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\omega = \sum dx_i \wedge dy_i.
\]

There are no local invariants (like curvature).
Compact examples

- Even-dimensional spheres $S^{2n} = \{ \overline{x} \in \mathbb{R}^{2n+1} \mid \sum x_i^2 = 1 \}$
- Complex projective space $\mathbb{C}P^{n-1}$
- Grassmannian $\mathcal{G}r(k, \mathbb{C}^n)$
- Flag varieties $\mathcal{F}lags(\mathbb{C}^n)$
- Complex projective varieties
- Toric varieties
- Based loops
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- Complex projective varieties
- Toric varieties
- Based loops $\Omega G = \{ \gamma : S^1 \to G \mid \gamma(\text{Id}) = \text{Id} \}$
Example: $\mathcal{P}ol_d(a_1, \ldots, a_n)$

$\in \mathcal{P}ol_2(a_1, \ldots, a_5)$
Example: $\mathcal{Pol}_d(a_1, \ldots, a_n)$

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\[ \mathcal{P}ol_d(a_1, \ldots, a_n) = \{ (\vec{v}_1, \ldots, \vec{v}_n) \mid \vec{v}_i \in \mathbb{R}^d, |v_i| = a_i, \sum \vec{v}_i = \vec{0} \} \quad \text{SO}(d) \]

$\mathcal{P}ol_3(a_1, \ldots, a_n)$ is symplectic! N.B. $d=3!!$
Symplectic actions

Symplectic manifolds often exhibit symmetries, encoded by a group action. (It’s a hard topological question, “How many manifolds do or do not have symmetries?” ...)

DEF: A group action $G \subset M$ is symplectic if it preserves $\omega$; that is, $\tau_g^* \omega = \omega \forall g \in G$. 

$S^1 \subset S^2$ by rotation

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Vector fields

**DEF:** Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. Suppose $G \subset M$. For any $\xi \in \mathfrak{g}$, we may define a vector field on $M$ by,

$$\mathcal{X}_\xi(p) = \frac{d}{dt} \left[ \exp(t \xi) \cdot p \right] \bigg|_{t=0}.$$
DEF: Suppose $G \subset (M, \omega)$. We say the action is Hamiltonian if

$$\omega(X_\xi, \cdot) = d\phi_\xi \quad \forall \xi \in \mathfrak{g}.$$ 

Example: $S^1 \subset M = S^2 = \mathbb{CP}^1$

\[
\begin{align*}
\omega &= d\theta \wedge dh \\
X_\xi &= \frac{\partial}{\partial \theta} \\
\omega(X_\xi, \cdot) &= dh \\
\Rightarrow \quad \phi_\xi &= h
\end{align*}
\]
A non-Hamiltonian action

**DEF:** Suppose $G \subseteq (M, \omega)$. We say the action is **Hamiltonian** if

$$\omega(X_\xi, \cdot) = d\phi^\xi \quad \forall \xi \in \mathfrak{g}.$$

**Non-Example:** $S^1 \subseteq T^2 = S^1 \times S^1$ rotating the first factor.

$$\omega = dx \wedge dy$$

$$X_\xi = \frac{\partial}{\partial x}$$

$$\omega(X_\xi, \cdot) = dy$$

But $dy \in H^1(T^2; \mathbb{Z})$ is certainly not exact!
Actions on $\mathcal{P}ol_d(a_1, \ldots, a_n)$
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$T^2 \subset \mathcal{P}ol_3(a_1, \ldots, a_5)$ is Hamiltonian.
The Moment Map

The Hamiltonian assumption says
\[ \omega(X_\xi, \cdot) = d\phi^\xi \quad \forall \xi \in \mathfrak{g}. \]

**DEF:** Combining these for all \( \xi \in \mathfrak{g} \), we define the **moment map**
\[
\Phi : M \rightarrow \mathfrak{g}^* \\
p \mapsto \left( \begin{array}{c}
\mathfrak{g} \\
\xi \\
\rightarrow \\
\mathbb{R} \\
\phi^\xi(p)
\end{array} \right).
\]

**Convexity Theorem** [Atiyah, Guillemin-Sternberg]:
If \( T = (S^1)^d \subset (M, \omega) \) is Hamiltonian, \( \Phi(M) \) is a convex polytope.
\[ \Phi(M) = \text{Conv}(\Phi(M^T)). \]
Examples
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$\mathbb{C}P^3 \cong \mathcal{G}r(2, \mathbb{C}^4)$

$\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1 \cong \Omega SU(2)$

$\mathcal{P}ol_3(a_1, \ldots, a_5)$
Moment polytope geometry

The moment polytope has additional structure, coming from the orbits of the torus action.

\[
\begin{align*}
\{ \text{Vertices of } \Delta \} & \leftrightarrow \{ \text{T-fixed points} \} \\
\{ \text{Edges of } \Delta \} & \leftrightarrow \{ \text{Points fixed by some } S \cong T^{d-1} \} \\
\{ \text{k-faces of } \Delta \} & \leftrightarrow \{ \text{Points fixed by some } S \cong T^{d-k} \}
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The moment polytope is a discrete representation of the orbit space $M/T$. 
Chambers of the moment polytope

**DEF:** The chambers of the moment polytope are the open regions in

$$\Delta - \{(d - 1)\text{-dimensional facets}\}.$$  

**Questions:**

- Can we count the chambers?
- Can we distinguish the chambers?
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Chambers of the moment polytope

To any point $\mu$ in a chamber, we may associate an ideal

$$I_\mu \subseteq H^*_T(M).$$

**Theorem** [Goldin-H-Jeffrey]:
Suppose $T$ acts on $M$ Hamiltonianly with isolated fixed points, and $\mu_1$ and $\mu_2$ are elements of chambers of $\Delta$. Then

$$I_{\mu_1} = I_{\mu_2}$$

if and only if $\mu_1$ and $\mu_2$ are in the same chamber of $\Delta$. 
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The classes that distinguish chambers of the permutahedron are called permuted Schubert polynomials.
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Functor $\text{Spaces} \rightarrow \text{Rings}$
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Functor \( \text{Spaces} \rightarrow \text{Rings} \)

\[
G \mathcal{C} M \overset{\sim}{\longrightarrow} H^*_G(M; \mathbb{Z}) \text{ or } H^*_G(M; \mathbb{R})
\]

\( f : M \rightarrow N \implies f^* : H^*_G(N) \rightarrow H^*_G(M) \)

Mayer-Vietoris

Et cetera
Equivariant cohomology is a generalized cohomology theory in the equivariant category.

- **Functor** \( \text{Spaces} \rightarrow \text{Rings} \)
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 Functor \( \text{Spaces} \rightarrow \text{Rings} \)

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T = T^d = S^1 \times \cdots \times S^1 \leadsto H_T^*(\text{pt}; \mathbb{Z}) = \mathbb{Z}[x_1, \ldots, x_d] \\
\text{deg}(x_i) = 2
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$H_T^*(pt; \mathbb{R}) = \mathbb{R}[x_1, \ldots, x_d] \cong \text{Sym}(t^*)$
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\[ G \subset M \xrightarrow{\sim} H^*_G(M; \mathbb{Z}) \text{ or } H^*_G(M; \mathbb{R}) \]
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- Equivariant cohomology of a point is not \( \mathbb{Z} \)
- Spaces, maps should be equivariant
- If \( G \curvearrowright M \) is a free action, then \( H^*_G(M) = H^*(M/G) \)
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- Equivariant cohomology of a point is not $\mathbb{Z}$
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- If $G \subset C M$ is a free action, then $H^*_G(M) = H^*(M/G)$
Cohomological Localization

\[ M^T \xrightarrow{\sim} M \quad \xrightarrow{\sim} \quad H^*_T(M; \mathbb{R}) \rightarrow H^*_T(M^T; \mathbb{R}) \]

**Borel’s Theorem:**
The kernel and cokernel of the map

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are torsion submodules (over \( H^*_T(pt; \mathbb{Q}) \)).
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**Theorem** [Frankel, Atiyah, Kirwan]:
If \( T^C M \) is a compact Hamiltonian \( T \)-manifold, then

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**Theorem** [Frankel, Atiyah, Kirwan]:
If \( T \subseteq M \) is a compact Hamiltonian \( T \)-manifold, then
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is an injection. *(The statement sometimes holds over \( \mathbb{Z} \).)*
Example

\[ T^3 \subset Gr(2, \mathbb{C}^4) \sim \sim \to H^*_T(Gr(2, \mathbb{C}^4); \mathbb{Z}) \subseteq \bigoplus_{i=1}^{6} \mathbb{Z}[x, y, z] \]
Example

$T^3 \subset Gr(2, \mathbb{C}^4) \xrightarrow{\Phi} H_T^*(Gr(2, \mathbb{C}^4); \mathbb{Z}) \subseteq \bigoplus_{i=1}^{6} \mathbb{Z}[x, y, z]$
Equivariant cohomology

\[ M^T \xrightarrow{\sim} M \quad \xrightarrow{\sim} \quad H^*_T(M; R) \rightarrow H^*_T(M^T; R) \]

Equivariant cohomology

\[ \mathcal{M}^T \rightarrow \mathcal{M} \quad \overset{\sim}{\longrightarrow} \quad H_T^*(\mathcal{M}; R) \rightarrow H_T^*(\mathcal{M}^T; R) \]


\[ \alpha \in H_T^*(\mathcal{M}; R) \quad \mapsto \quad (\alpha|_N, \alpha|_S) \in \mathbb{R}[x] \oplus \mathbb{R}[x] \]
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\[ \alpha \in H^*_T(M; R) \implies (\alpha|_N, \alpha|_S) \in \mathbb{R}[x] \oplus \mathbb{R}[x] \]

\[ \int_M \alpha = \sum_{F \subseteq M^T} \int_F \frac{\alpha|_F}{e_T(\nu(F \subseteq M))} \implies \frac{\alpha|_N}{x} + \frac{\alpha|_S}{-x} \in \mathbb{R}[x] \]

\[ \iff x \mid (\alpha|_N - \alpha|_S) \]
Equivariant cohomology

\[ M^T \hookrightarrow M \quad \sim \sim \quad H^*_T(M; R) \rightarrow H^*_T(M^T; R) \]

Equivariant cohomology

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Equivariant cohomology

\[
\mathcal{M}^T \xrightarrow{\sim} \mathcal{M} \xrightarrow{\sim} H_\mathcal{T}^*(\mathcal{M}; R) \rightarrow H_\mathcal{T}^*(\mathcal{M}^T; R)
\]


Equivariant cohomology of $\Omega G$


**Theorem** [Harada-Henriques-H.]:
The GKM description works, even in infinite dimensional cases, for

- Equivariant cohomology $H^*_T(M; \mathbb{Q})$
  (Sometimes integrally!) $H^*_T(M; \mathbb{Z})$
- Equivariant K-theory $K^*_T(M)$
- Equivariant cobordism $MU^*_T(M)$
Equivariant cohomology of $\Omega G$


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- (Sometimes integrally!) $H_T^*(M; \mathbb{Z})$
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Theorem [Pabiniak]:
When a Hamiltonian circle action has isolated fixed points, and if there is a basis of $H^*_T(M; \mathbb{Q})$, then the integration formula gives a set of relations that describe

$$H^*_T(M; \mathbb{Q}) \subseteq H^*_T(M^T; \mathbb{Q}).$$
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Example:
There is a Hamiltonian circle action on $\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$, with 8 isolated fixed points. The moment map image is
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1 & \quad 2 & \quad 3 & \quad 4 & \quad 5 & \quad 6 & \quad 7 & \quad 8
\end{align*}$$

The relations: 

$$\begin{align*}
(f_i - f_j) & \in x \cdot \mathbb{Q}[x] \quad \forall \ i, j \\
f_4 - f_5 - f_6 - f_7 + 2f_8 & \\
f_3 - f_4 - f_5 + f_6 & \\
f_2 - f_3 - f_6 + f_7 & \\
2f_1 - f_2 - f_3 - f_4 + f_5 & 
\end{align*}$$

\[ \in x^2 \cdot \mathbb{Q}[x] \]
When the GKM description does not work ...

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\[
\left\{ \begin{align*}
  f_4 - f_5 - f_6 - f_7 + 2f_8 \\
  f_3 - f_4 - f_5 + f_6 \\
  f_2 - f_3 - f_6 + f_7 \\
  2f_1 - f_2 - f_3 - f_4 + f_5 \\
  2f_1 - f_2 - f_3 - 2f_4 + 2f_5 + f_6 + f_7 - 2f_8
\end{align*} \right\} \in x^2 \cdot \mathbb{Q}[x]
\]

\[
2f_1 - f_2 - f_3 - 2f_4 + 2f_5 + f_6 + f_7 - 2f_8 \in x^3 \cdot \mathbb{Q}[x]
\]
Cohomology of quotients

To any point $\mu$ in a chamber, the submanifold

$$\Phi^{-1}(\mu) \subset M$$

is $T$-invariant. If the $T$-action is free,

$$M/\!/T(\mu) = \Phi^{-1}(\mu)/T$$

is again a symplectic manifold called the **symplectic quotient**.
Cohomology of quotients

To any point \( \mu \) in a chamber, the submanifold

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Theorem [Kirwan]:

The inclusion \( \Phi^{-1}(\mu) \hookrightarrow M \) induces a surjection

\[ \kappa_\mu : H^*_T(M; \mathbb{R}) \to H^*(M//T(\mu); \mathbb{R}). \]
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The inclusion \( \Phi^{-1}(\mu) \hookrightarrow M \) induces a surjection

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The kernel \( \text{ker}(\kappa_\mu) = I_\mu \).
Cohomology of quotients

**Theorem** [Kirwan]:
The inclusion $\Phi^{-1}(\mu) \hookrightarrow M$ induces a surjection

$$\kappa_\mu : H^*_T(M; \mathbb{R}) \to H^*(M//T(\mu); \mathbb{R}).$$

The kernel $\ker(\kappa_\mu) = I_\mu$.

**Comments:**
- The kernel is often combinatorially computable
  [Tolman-Weitsman; Goldin]
- The ring $H^*(M//T(\mu); \mathbb{R})$ does NOT distinguish the chambers!
- A similar result is known in K-theory
  [Harada-Landweber]
- Similar results can be formulated for stringy invariants
Theorem [Hausmann-Faber-Schütz]:

When $M = \mathcal{G}r(2, \mathbb{C}^d)$, the ring $H^*(M//T(\mu); \mathbb{R})$ DOES distinguish the chambers, for $d \geq 5$. 
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N.B. $\mathcal{G}r(2, \mathbb{C}^d)//T(\mu) \cong \mathcal{P}ol(\alpha_\mu)$
Chambers and polygon spaces

**Theorem** [Hausmann-Faber-Schütz]:
When $M = \mathcal{G}r(2, \mathbb{C}^d)$, the ring $H^*(M//T(\mu); \mathbb{R})$ DOES distinguish the chambers, for $d \geq 5$. (Up to a natural action of $S_d$.)

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**Comments:**
- The proof is algebraic, passing to the real points
- Does a similar result hold for $\mathcal{G}r(k, \mathbb{C}^d)$, for some values of $d$?
- How does this relate to these questions for $\mathcal{F}lags(\mathbb{C}^d)$?