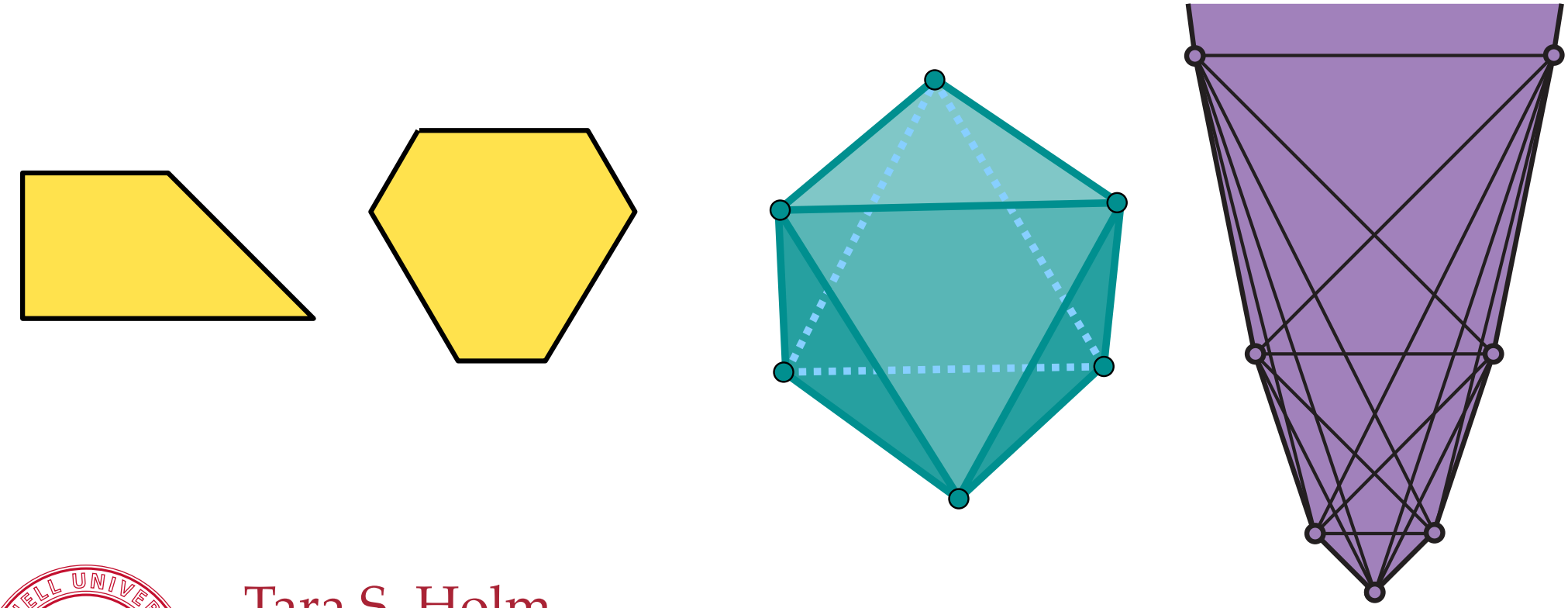
The Discrete Geometry of Moment Polytopes



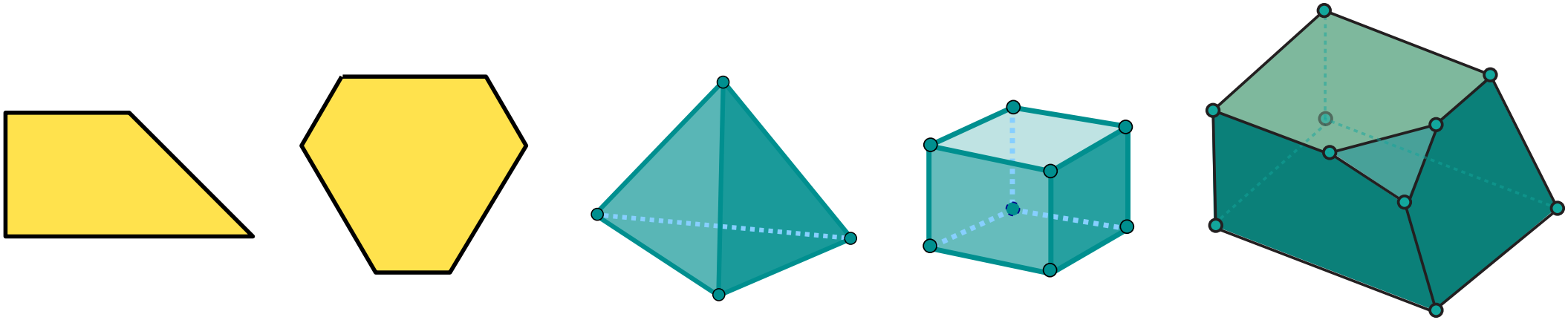
Tara S. Holm
Cornell University

FPSAC -- 3 August 2010

Slides made with Keynote - in PDF, actions do not work.

Simple polytopes

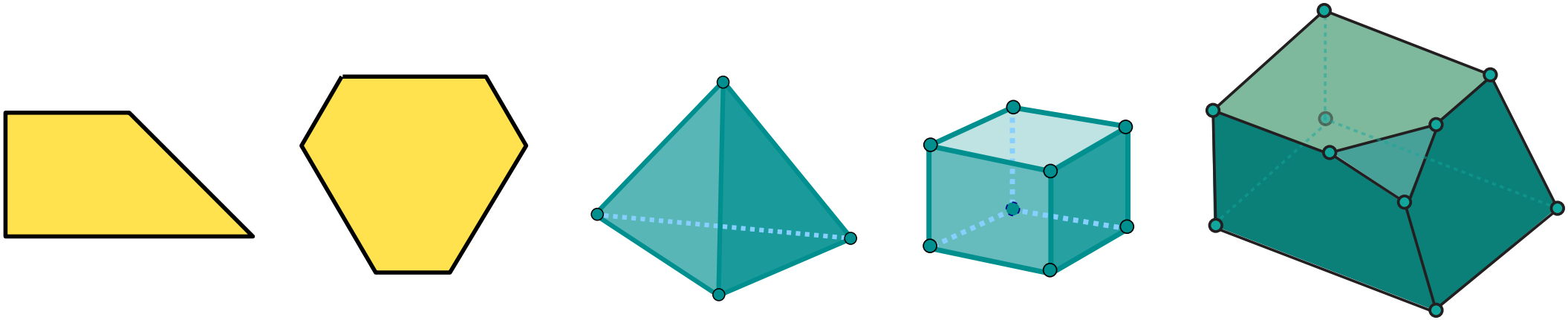
$\Delta \subseteq \mathbb{R}^d$



Simple polytopes

$$\Delta \subseteq \mathbb{R}^d \rightsquigarrow \mathbf{f} = (f_{-1}, f_0, f_1, \ldots, f_{d-1})$$

f_k = # k-dimensional faces



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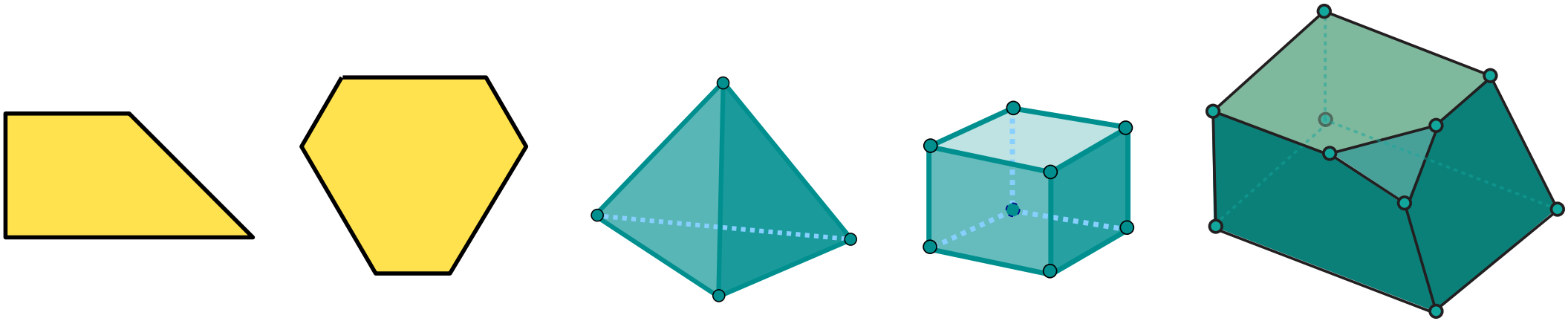

$\mathbf{f} = (1,4,4)$ $(1,6,6)$ $(1,4,6,4)$ $(1,8,12,6)$ $(1,10,15,7)$

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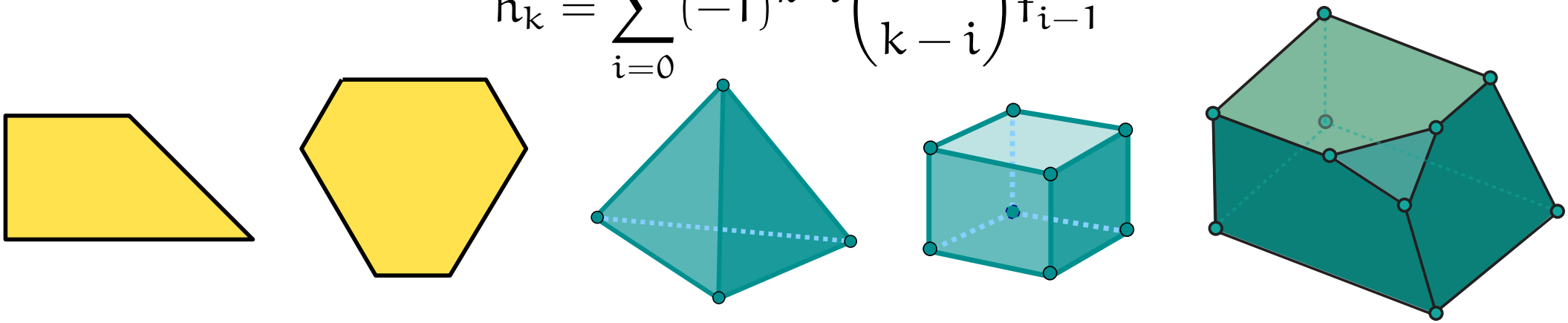

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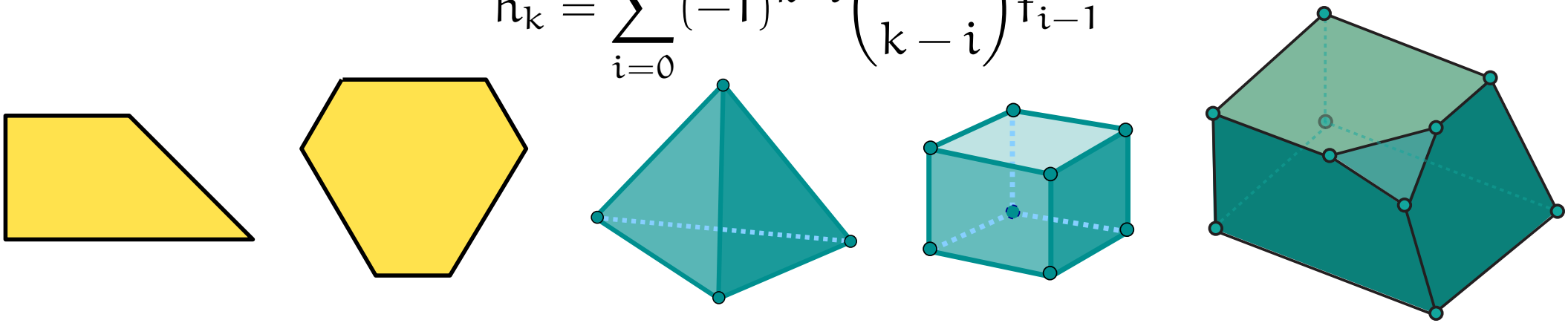

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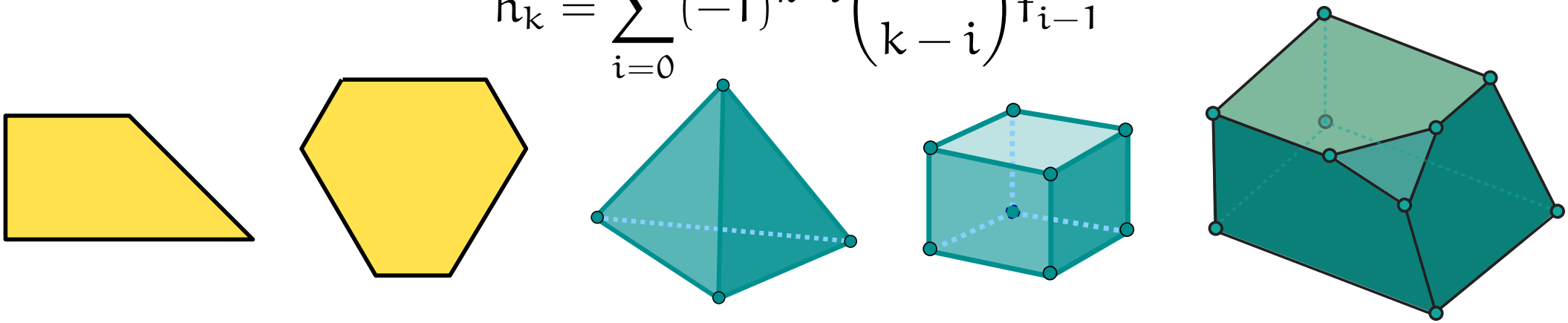

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<u>Theorem</u> [Stanley, 1975]:

(a) $h_k = h_{d-k}$

(b) $h_0 \leq h_1 \leq \cdots \leq h_{\lfloor \frac{d}{2} \rfloor}$

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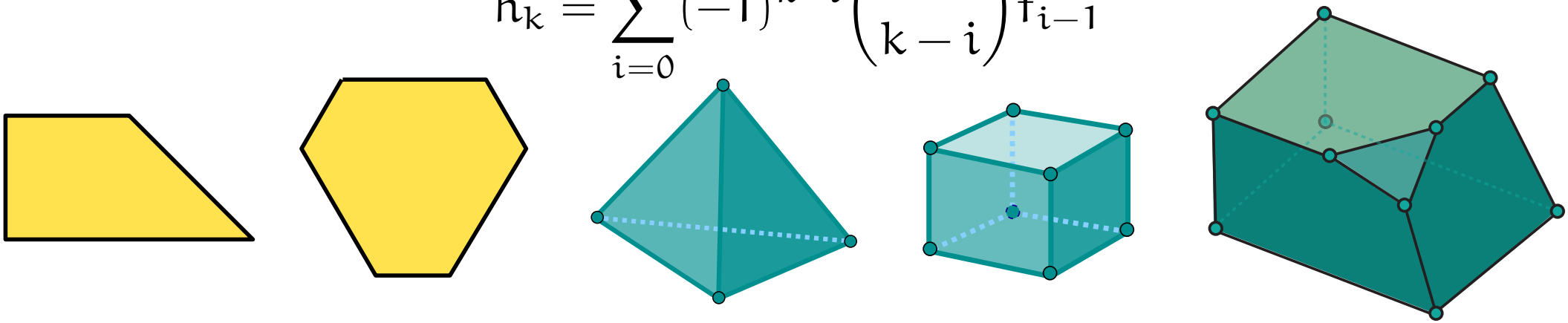

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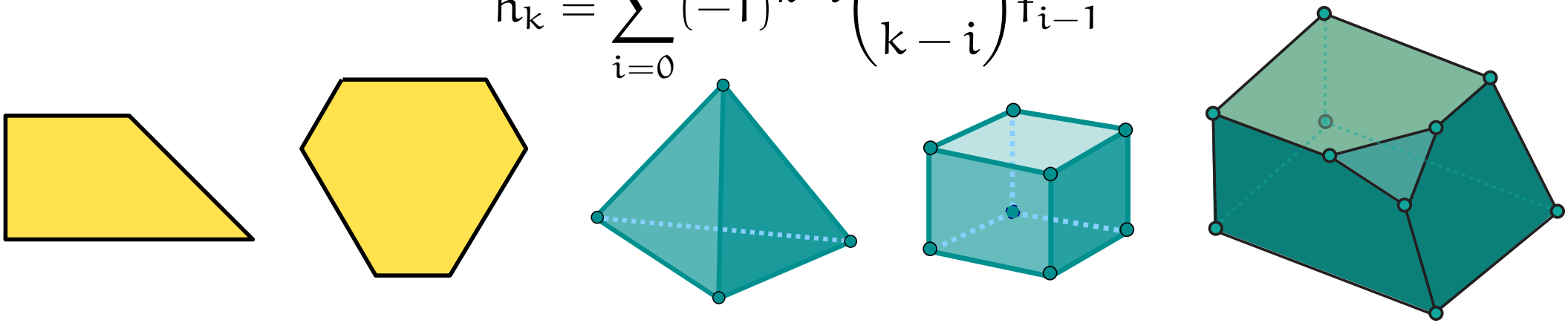

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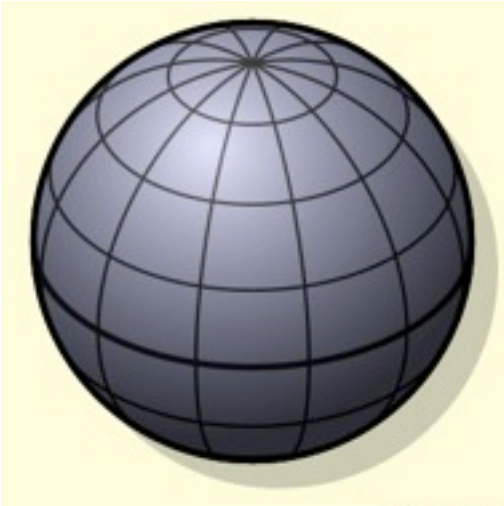
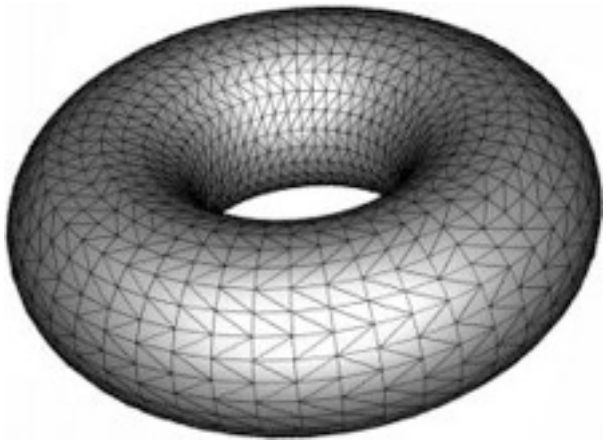
Symplectic manifolds

A symplectic manifold is a manifold

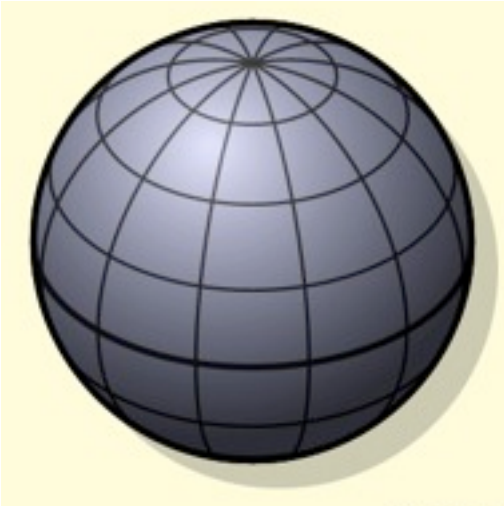


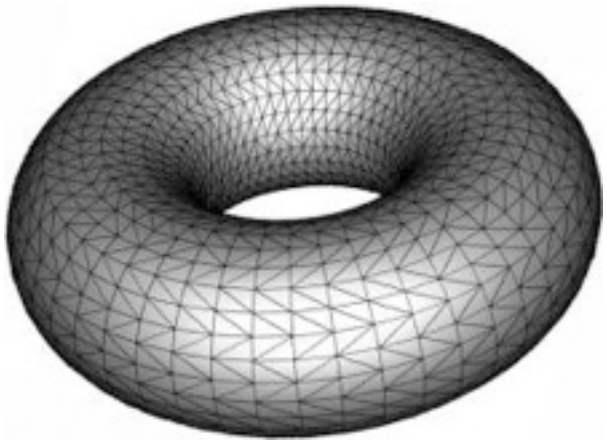


Symplectic manifolds

A symplectic manifold is a manifold with a two-form $\omega \in \Omega^2(M)$ that is:

- Closed: $d\omega = 0$
- Non-degenerate: $\omega^n = \mathrm{dVol} \rightsquigarrow$ M is $2n$-dimensional





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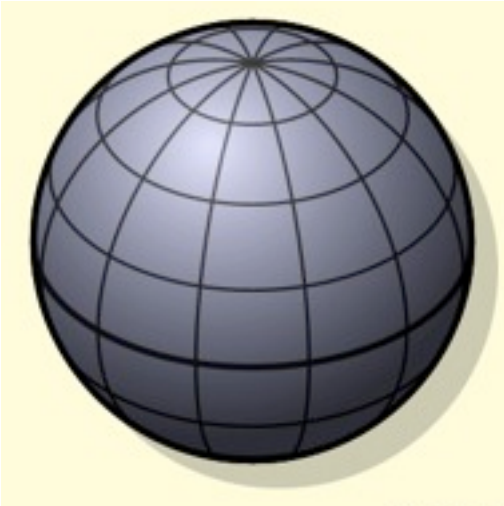

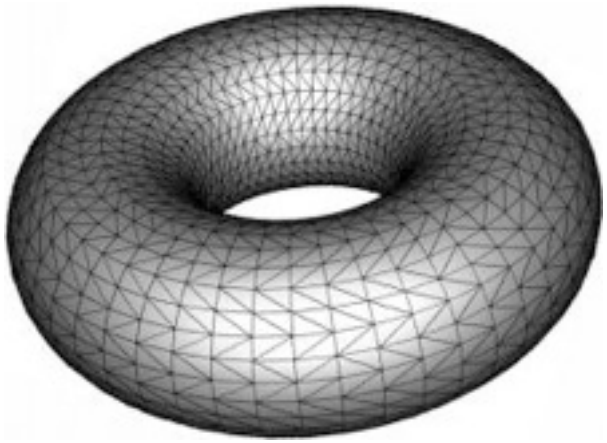

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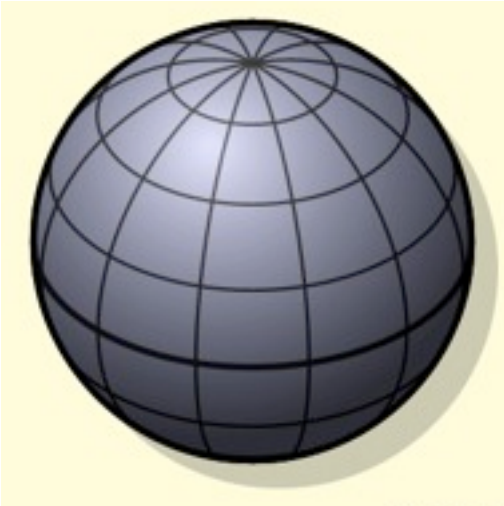

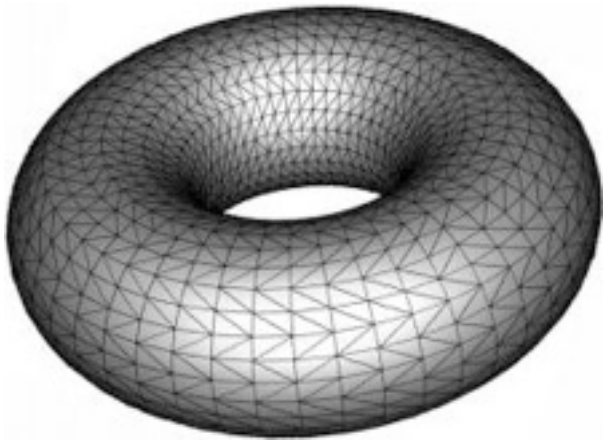

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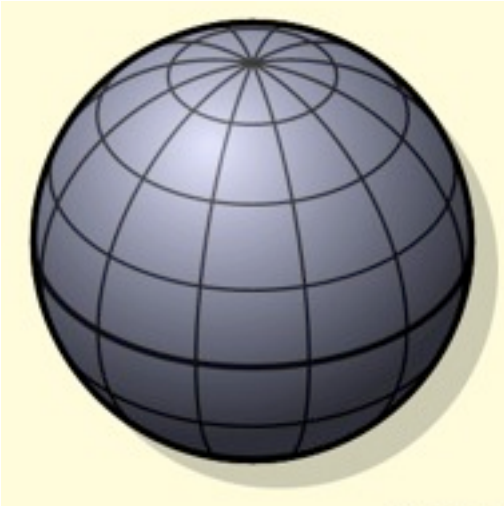

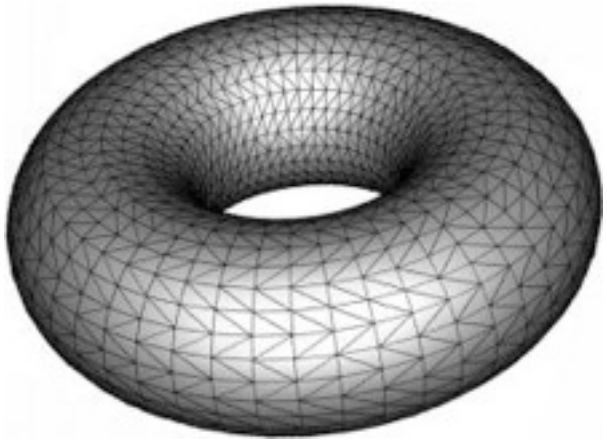

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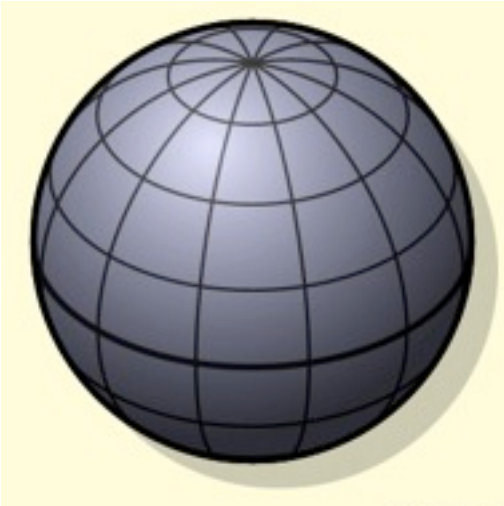

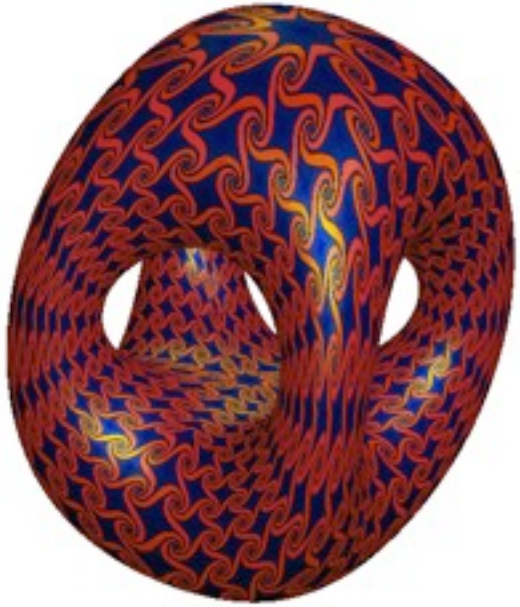

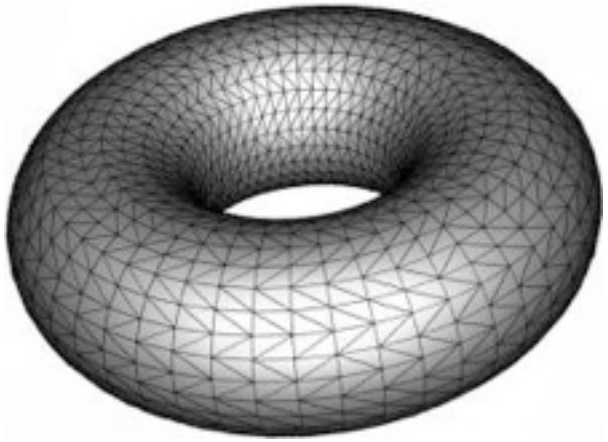

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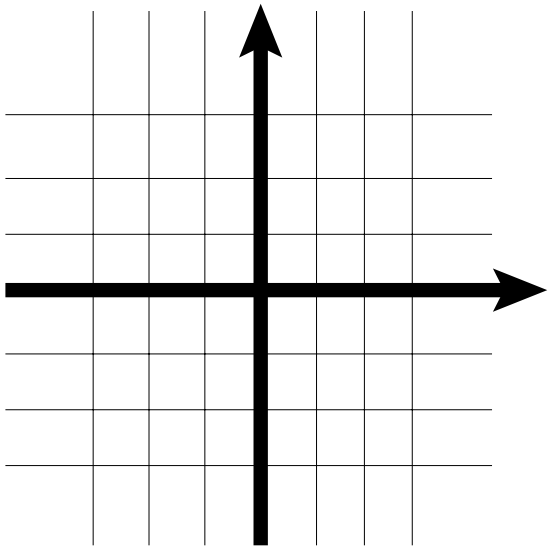

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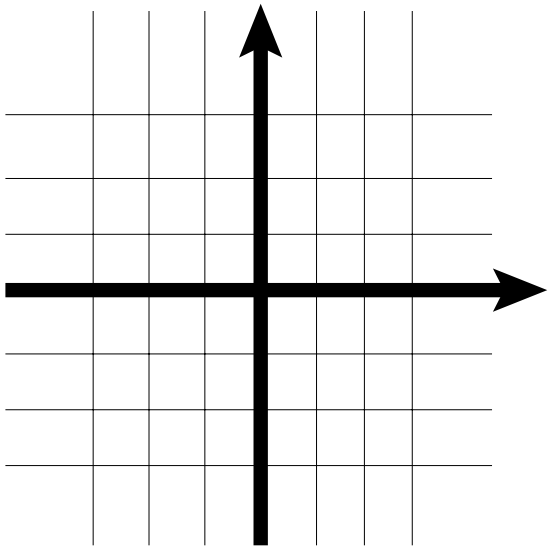

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<u>Darboux's Theorem</u>:

We may always choose coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n$ on M so that locally

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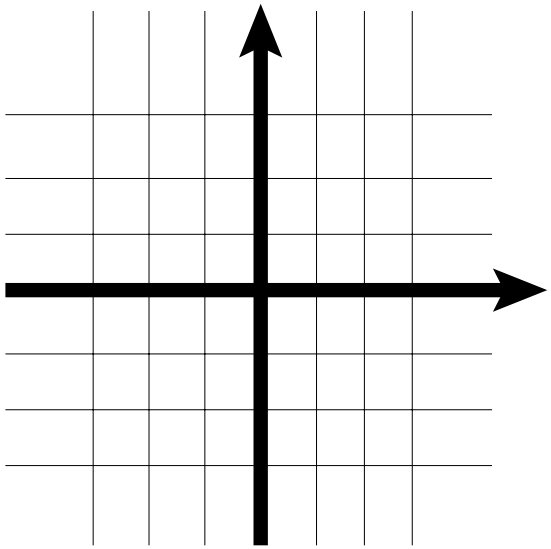

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$\rightsquigarrow$ There are no local invariants (like curvature).

Compact examples

- Even-dimensional spheres $S^{2n} = \left\{ \overrightarrow{x} \in \mathbb{R}^{2n+1} \;\middle|\; \sum x_i^2 = 1 \right\}$
- Complex projective space $\mathbb{C}P^{n-1}$
- Grassmannian $\mathcal{G}r(k, \mathbb{C}^n)$
- Flag varieties

  $\mathcal{F}\ell ags(\mathbb{C}^n)$
- Complex projective varieties
- Toric varieties
- Based loops

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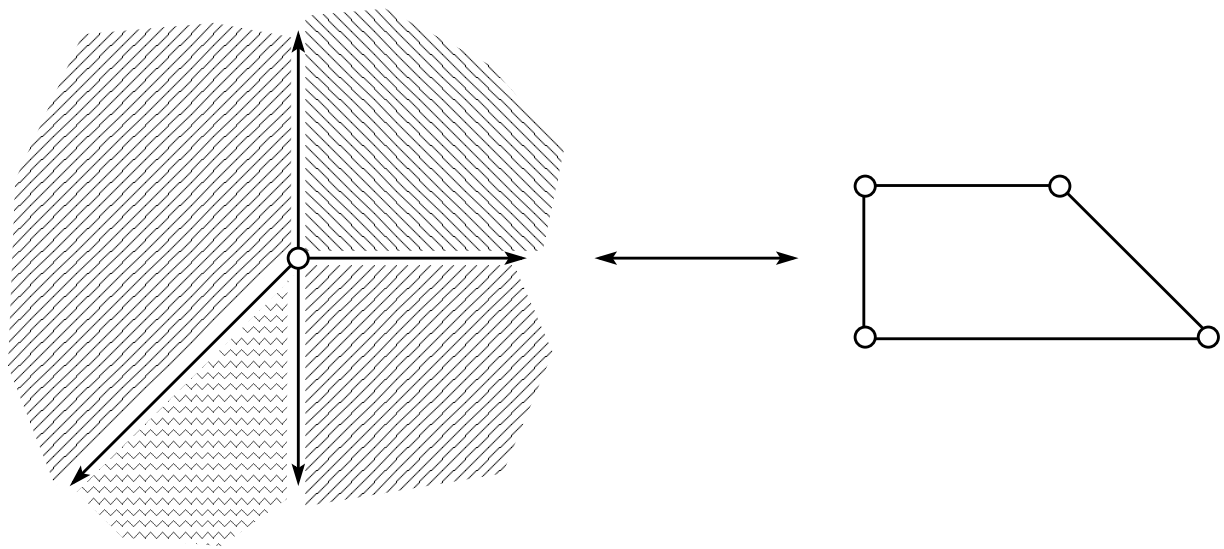

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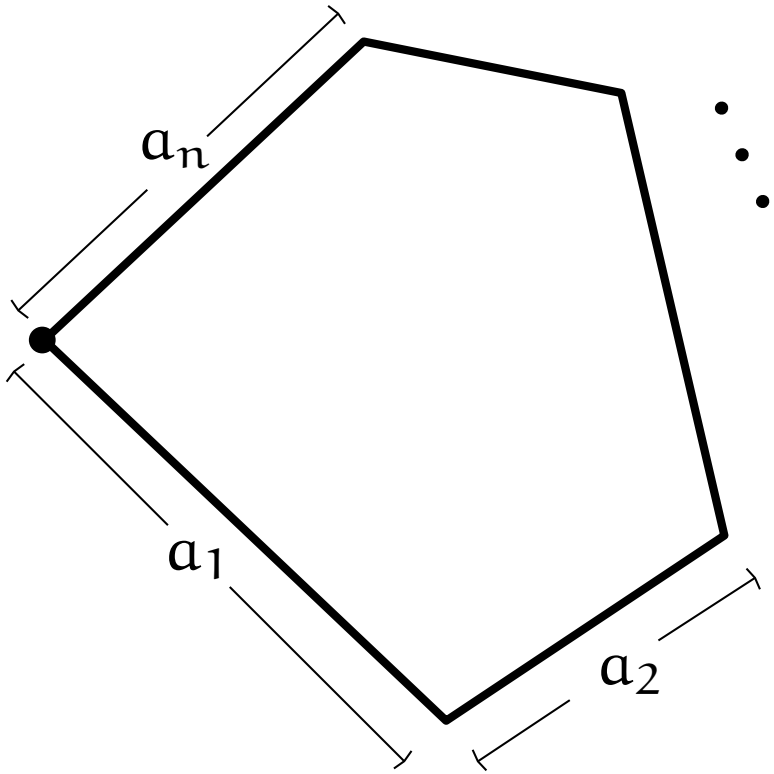
- Based loops $\Omega G = \{\gamma : S^1 \to G \mid \gamma(\mathrm{Id}) = \mathrm{Id}\}$

Example: $\mathcal{P}ol_d(a_1, \ldots, a_n)$



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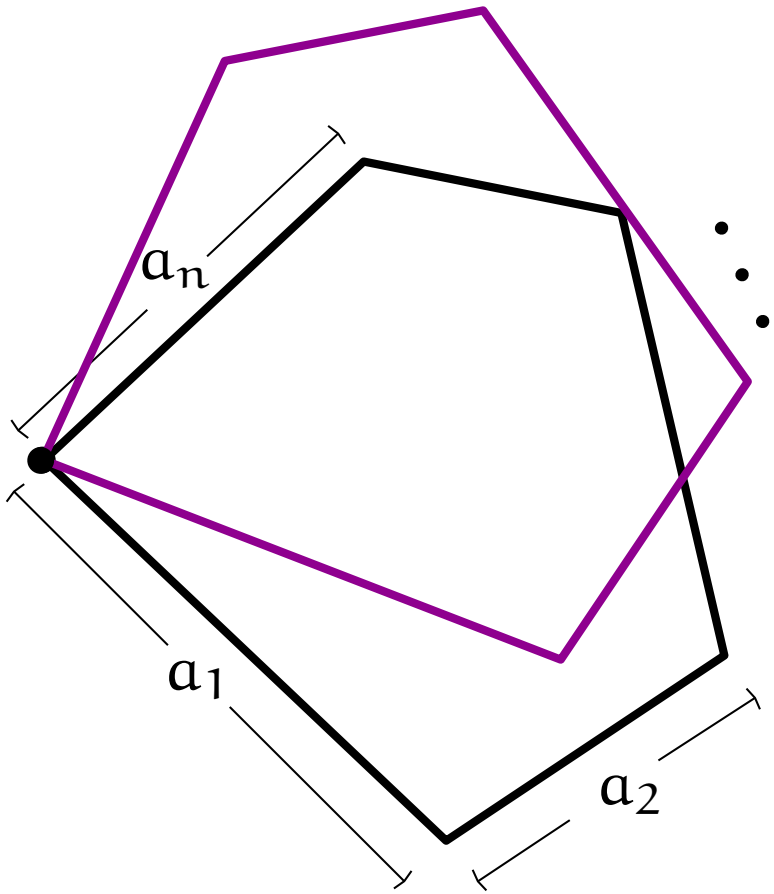

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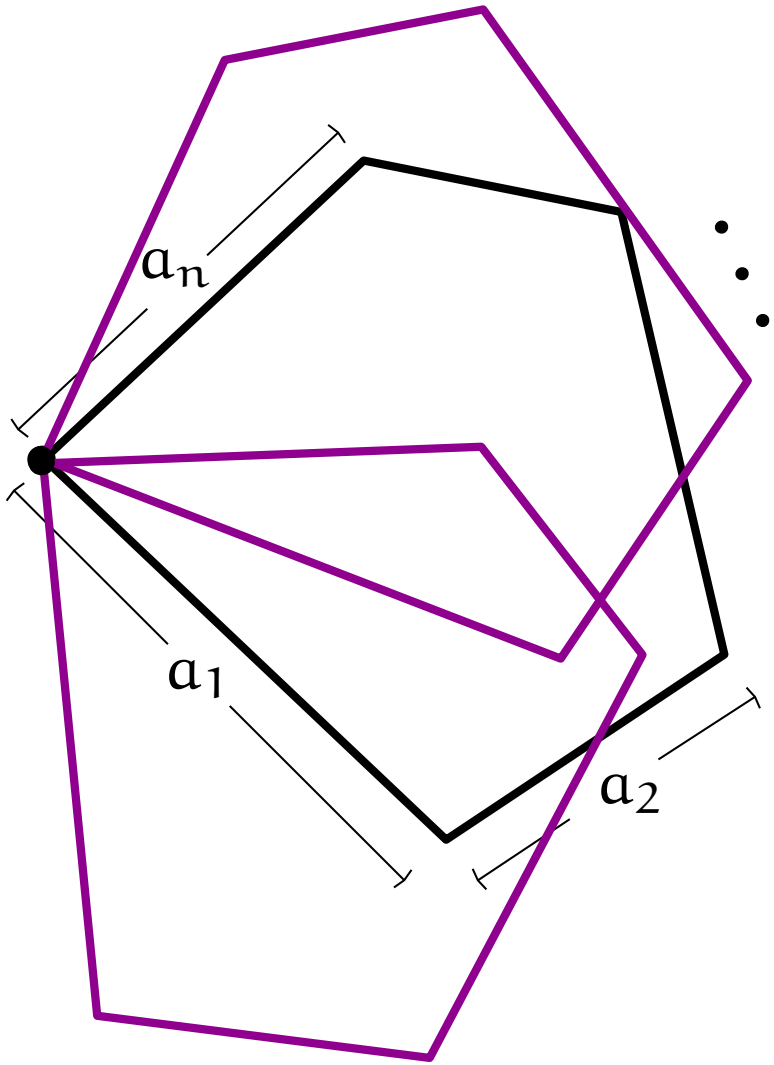

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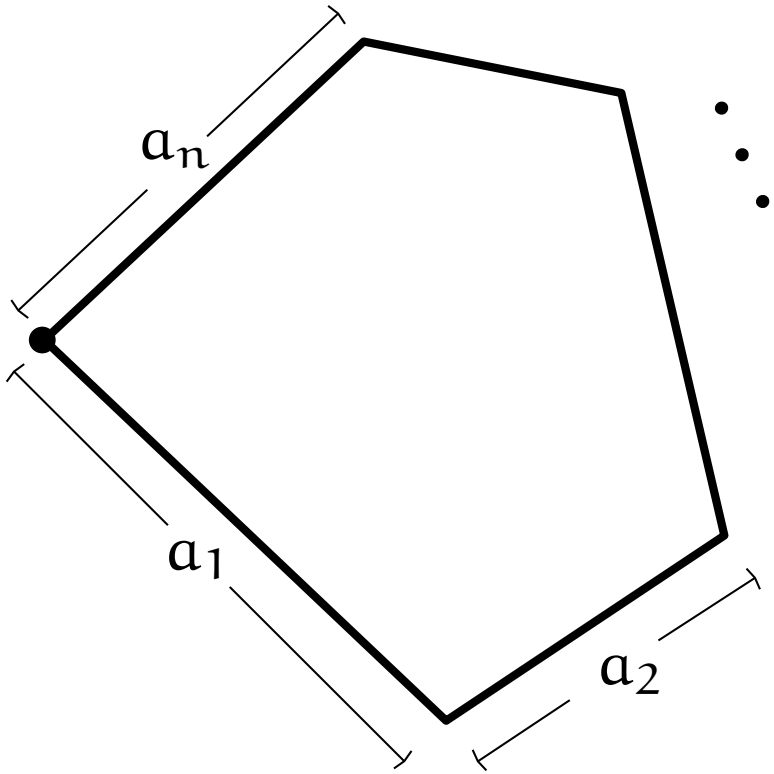


$\in \mathcal{P}ol_2(a_1, \ldots, a_5)$

$$\mathcal{P}ol_d(a_1, \ldots, a_n) = \frac{\left\{ (\overrightarrow{v_1}, \ldots, \overrightarrow{v_n}) \;\middle|\; \overrightarrow{v_i} \in \mathbb{R}^d,\ |v_i| = a_i,\ \sum \overrightarrow{v_i} = \overrightarrow{0} \right\}}{SO(d)}$$

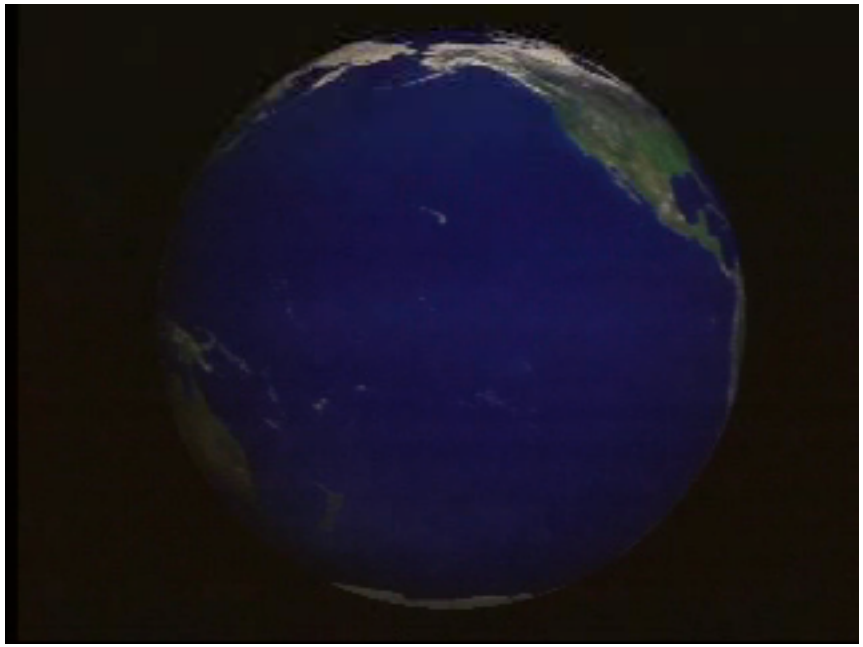
$\mathcal{P}ol_3(a_1, \ldots, a_n)$ is symplectic! N.B. d=3!!

Symplectic actions

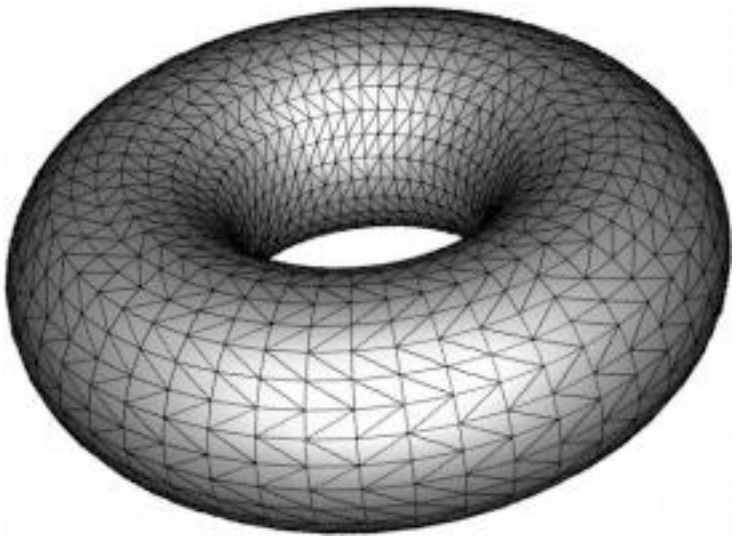
Symplectic manifolds often exhibit symmetries, encoded by a group action. (It's a hard topological question, "How many manifolds do or do not have symmetries?" ...)

DEF: A group action $G \circlearrowright M$ is symplectic if it preserves ω; that is,

$$\tau_g^* \omega = \omega \quad \forall\, g \in G.$$



$S^1 \circlearrowright S^2$ by rotation



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Symplectic actions

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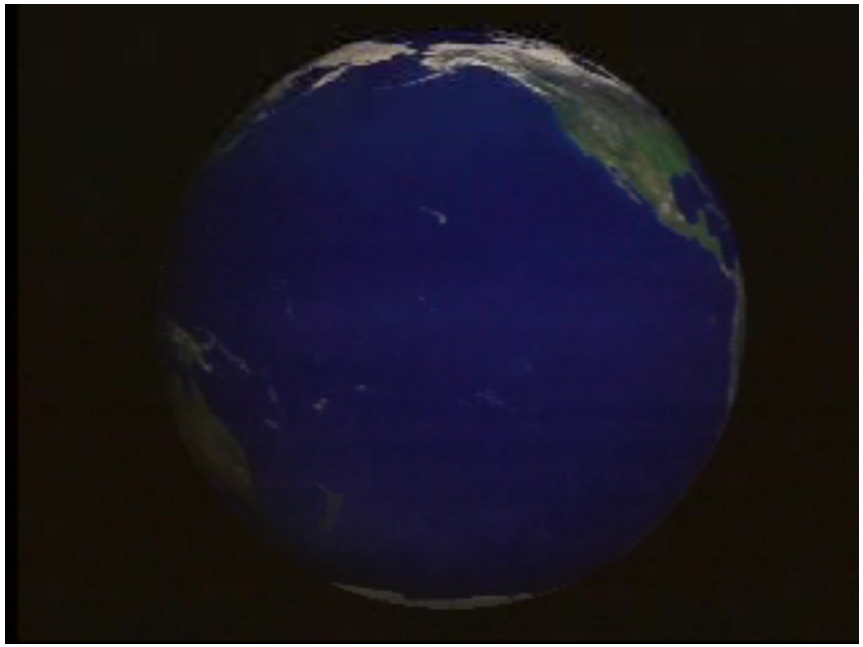

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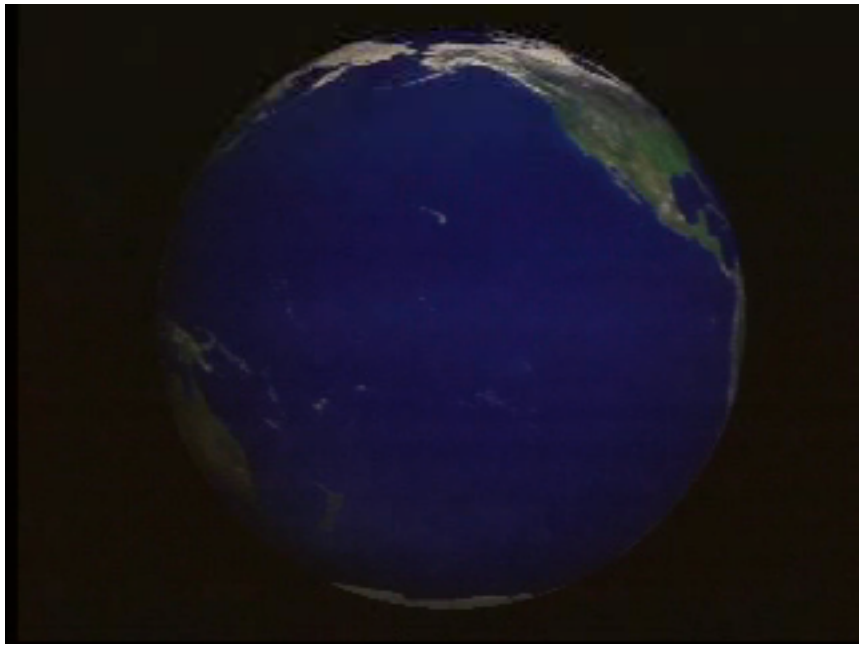

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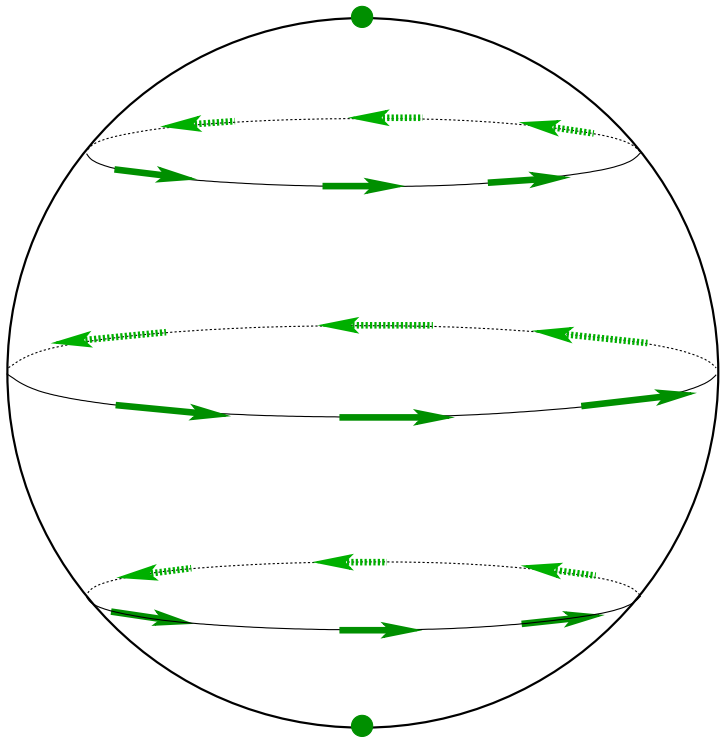
Vector fields

DEF: Let G be a Lie group with Lie algebra $\mathfrak{g}$. Suppose $G \circlearrowright M$. For any $\xi \in \mathfrak{g}$, we may define a vector field on M by,

$$\mathcal{X}_\xi(p) = \frac{d}{dt}\Big[\exp(t\xi)\cdot p\Big]\Big|_{t=0}.$$



$S^1 \circlearrowright S^2$ by rotation

$\rightsquigarrow$ Vector field parallel to latitude lines

Hamiltonian actions

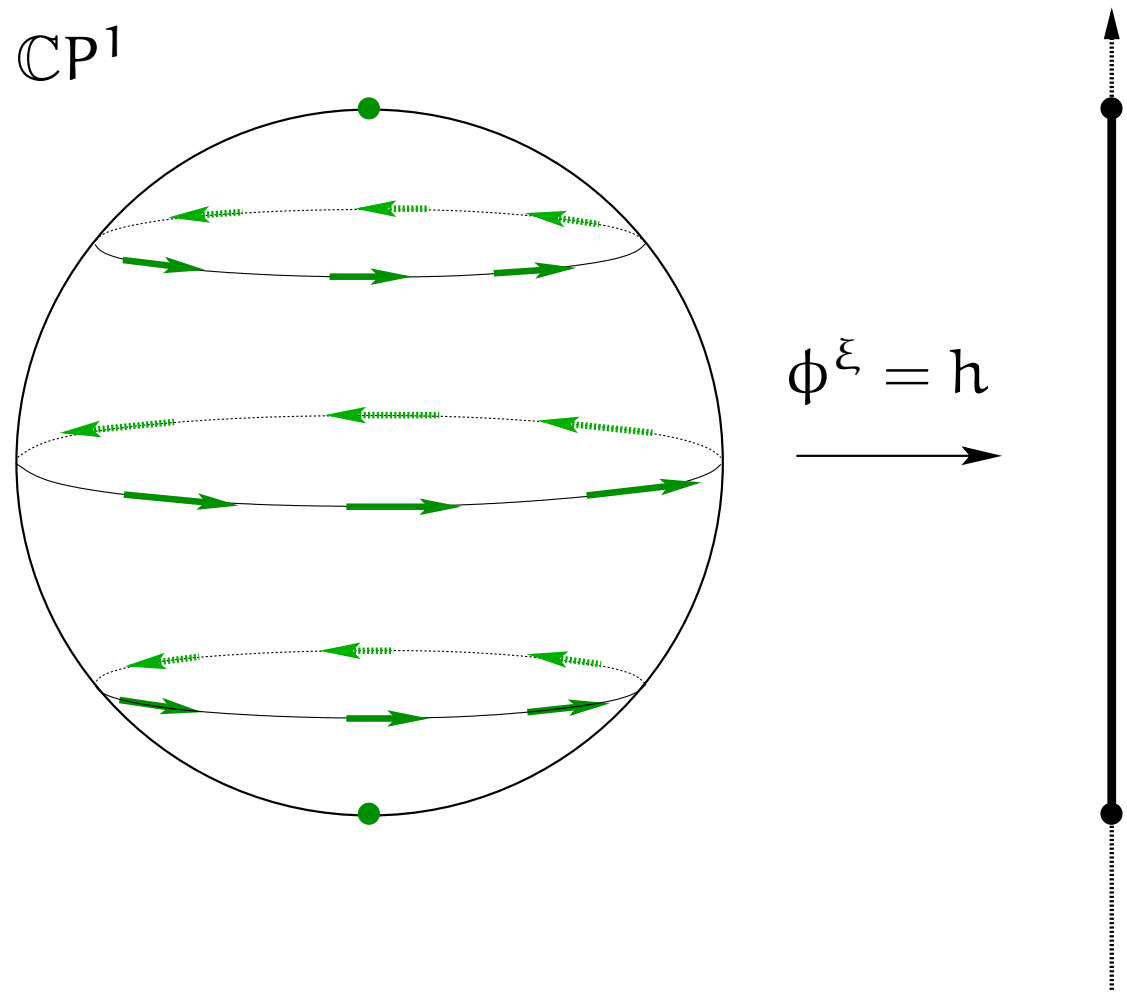
DEF: Suppose $G \circlearrowright (M, \omega)$. We say the action is Hamiltonian if

$$\omega(X_\xi, \cdot) = d\phi^\xi \quad \forall\, \xi \in \mathfrak{g}.$$

Example: $S^1 \circlearrowright M = S^2 = \mathbb{C}P^1$

$$\begin{aligned} \omega &= d\theta \wedge dh \\ \mathcal{X}_\xi &= \frac{\partial}{\partial \theta} \\ \omega(\mathcal{X}_\xi, \cdot) &= dh \end{aligned}$$

$$\Longrightarrow \quad \phi^\xi = h$$



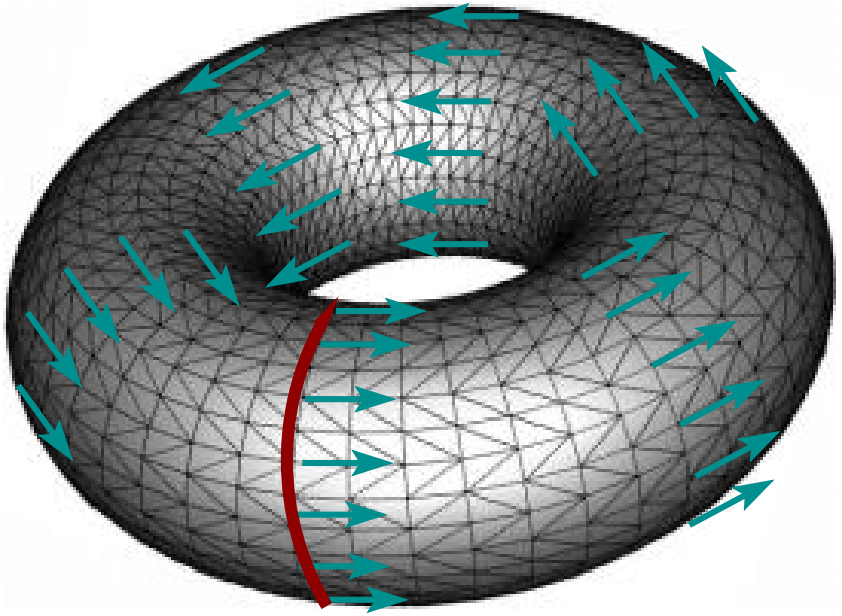
A non-Hamiltonian action

DEF: Suppose $G \circlearrowright (M, \omega)$. We say the action is Hamiltonian if

$$\omega(X_\xi, \cdot) = d\phi^\xi \quad \forall\, \xi \in \mathfrak{g}.$$

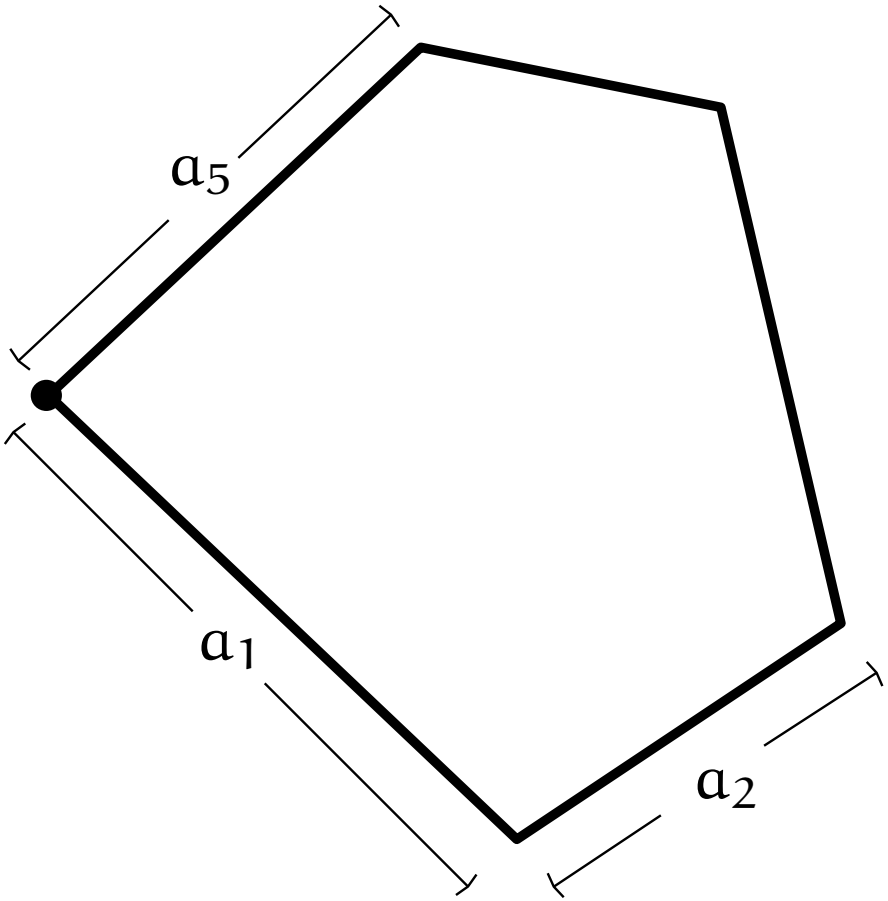
Non-Example: $S^1 \circlearrowright T^2 = S^1 \times S^1$ rotating the first factor.

$$\begin{aligned} \omega &= dx \wedge dy \\ \mathcal{X}_\xi &= \frac{\partial}{\partial x} \\ \omega(\mathcal{X}_\xi, \cdot) &= dy \end{aligned}$$



But $dy \in H^1(T^2; \mathbb{Z})$ is certainly not exact!

Actions on $\mathcal{P}ol_d(a_1, \ldots, a_n)$



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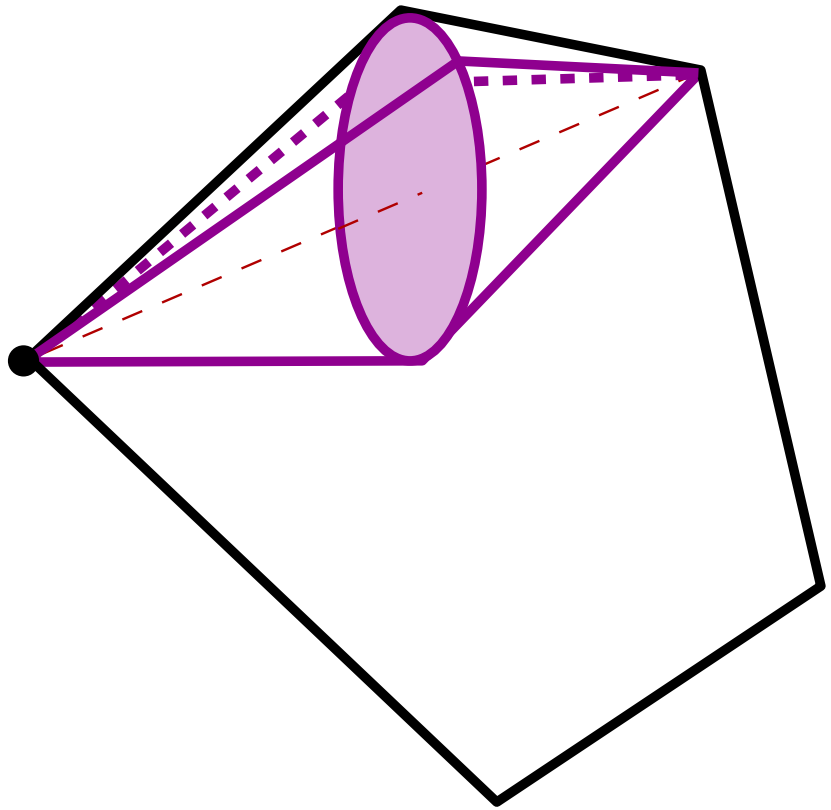

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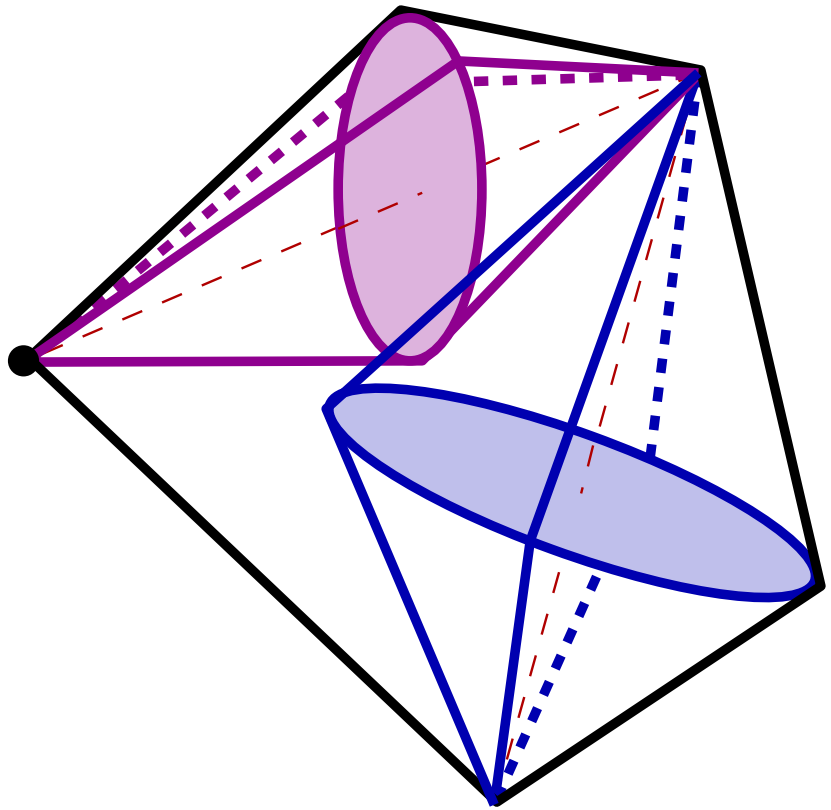

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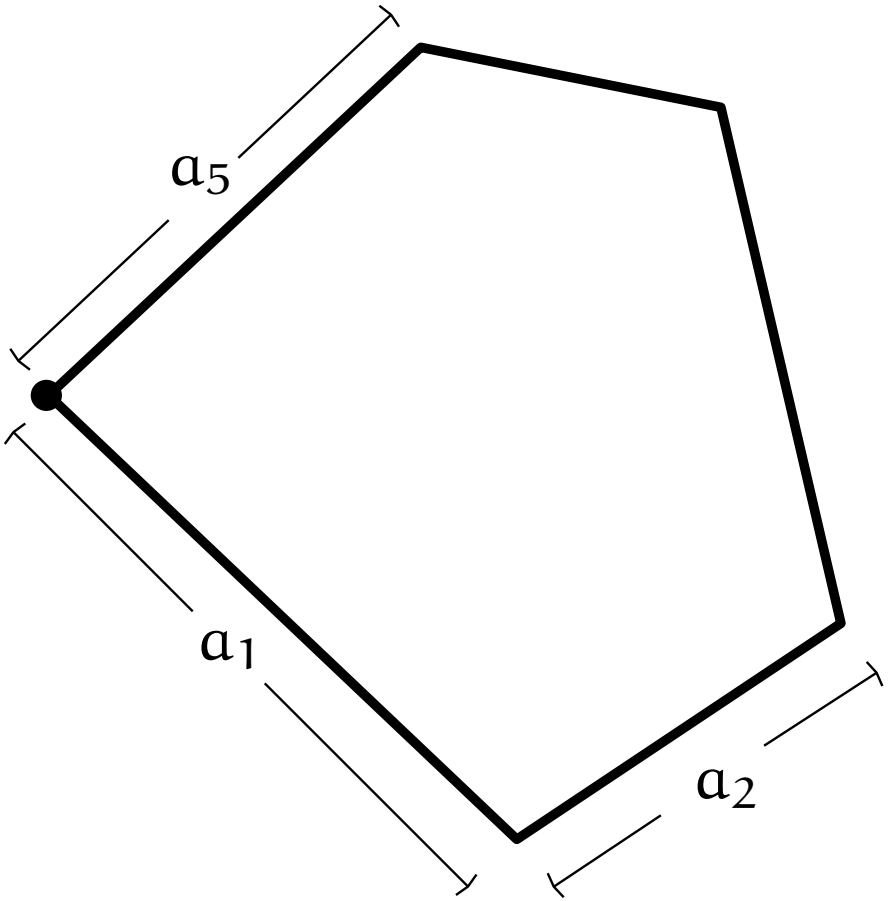


$T^2 \circlearrowright \mathcal{P}ol_3(a_1, \ldots, a_5)$ is Hamiltonian.

The Moment Map

The Hamiltonian assumption says

$$\omega(X_\xi, \cdot) = d\phi^\xi \quad \forall\, \xi \in \mathfrak{g}.$$

DEF: Combining these for all $\xi \in \mathfrak{g}$, we define the moment map

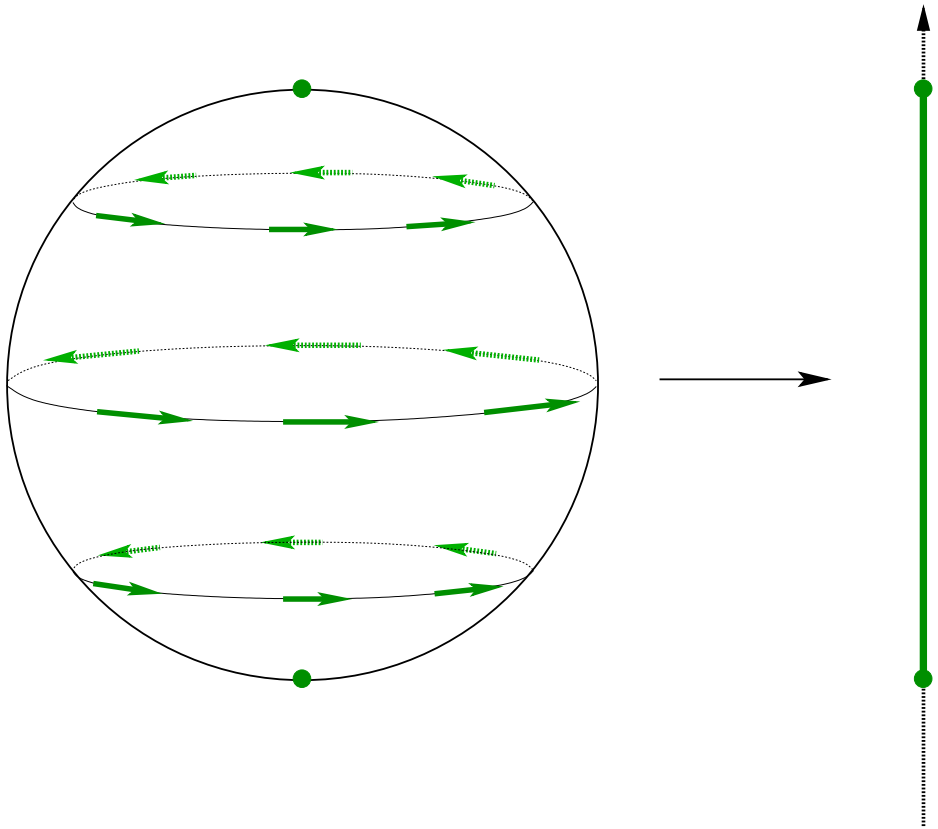
$$\begin{array}{rcl} \Phi : M & \to & \mathfrak{g}^* \\ p & \mapsto & \left(\begin{array}{ccc} \mathfrak{g} & \longrightarrow & \mathbb{R} \\ \xi & \mapsto & \phi^\xi(p) \end{array} \right). \end{array}$$

<u>Convexity Theorem</u> [Atiyah, Guillemin-Sternberg]:

If $T = (S^1)^d \circlearrowright (M, \omega)$ is Hamiltonian, $\Phi(M)$ is a convex polytope.

$$\Phi(M) = \mathrm{Conv}(\Phi(M^T)).$$

Examples



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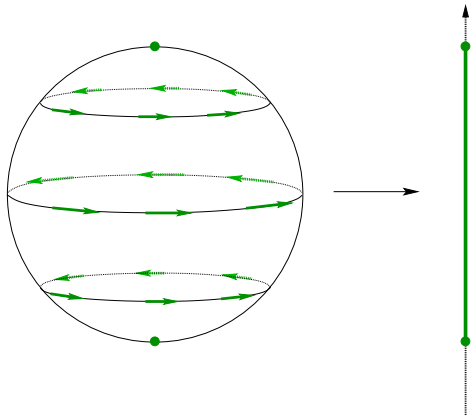
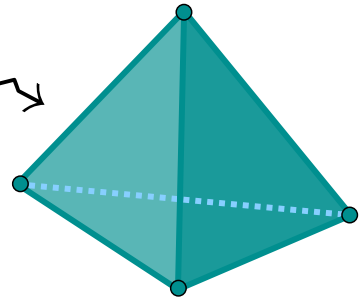
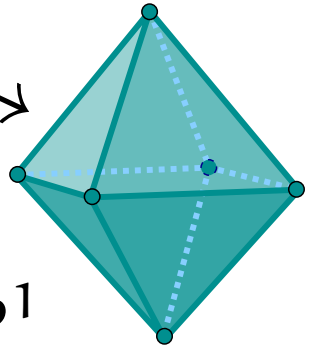


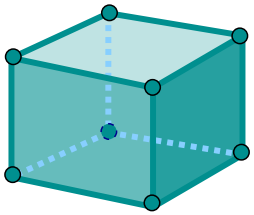
$\mathbb{C}P^3$



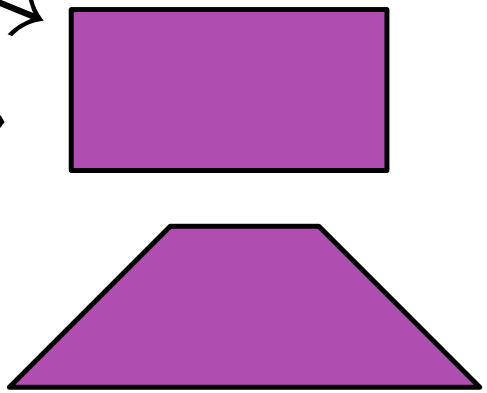
$\mathcal{G}r(2, \mathbb{C}^4)$



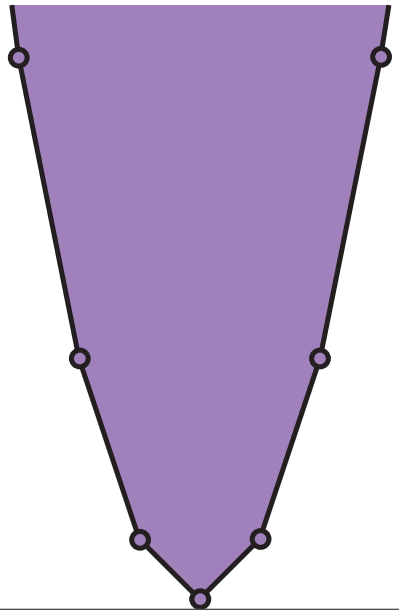
$\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$



$\mathcal{P}ol_3(a_1, \ldots, a_5)$



$\Omega SU(2)$



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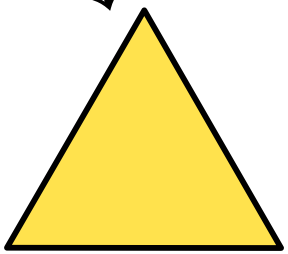
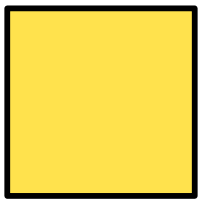
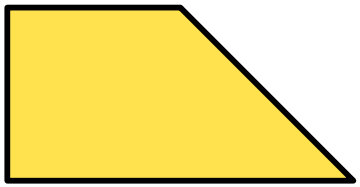
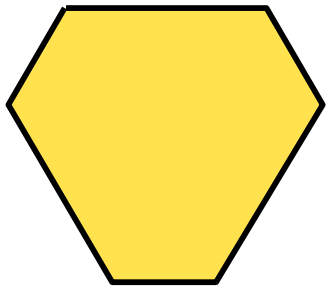


$\mathbb{C}P^1 \times \mathbb{C}P^1$







Moment polytope geometry

The moment polytope has additional structure, coming from the orbits of the torus action.

$$\left\{\text{Vertices of } \Delta\right\} \leftrightsquigarrow \left\{T\text{-fixed points}\right\}$$

$$\left\{\text{Edges of } \Delta\right\} \leftrightsquigarrow \left\{\text{Points fixed by some } S \cong T^{d-1}\right\}$$

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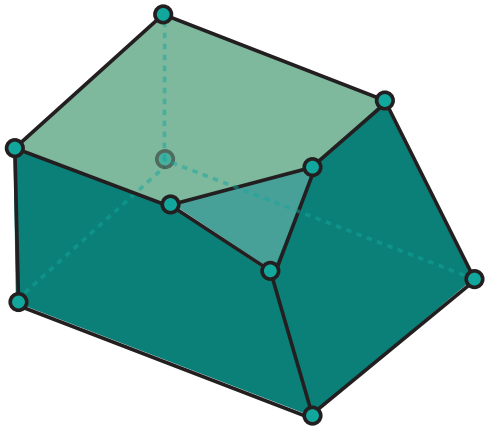

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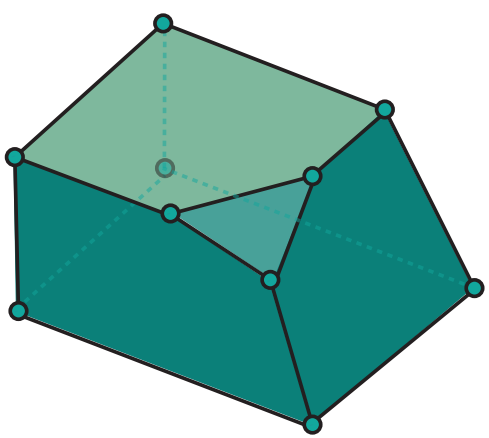

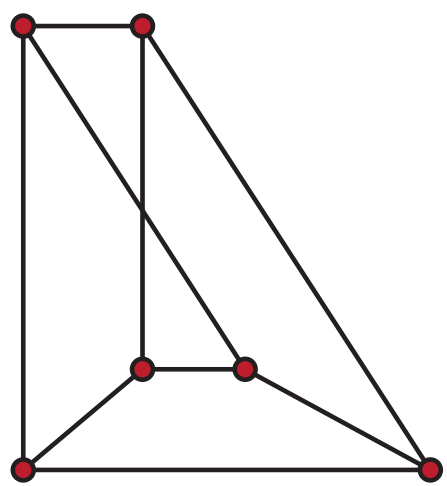

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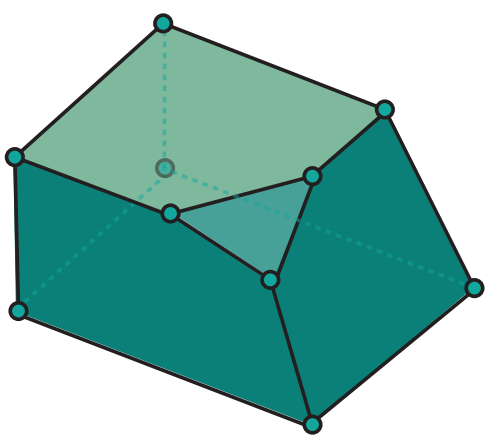

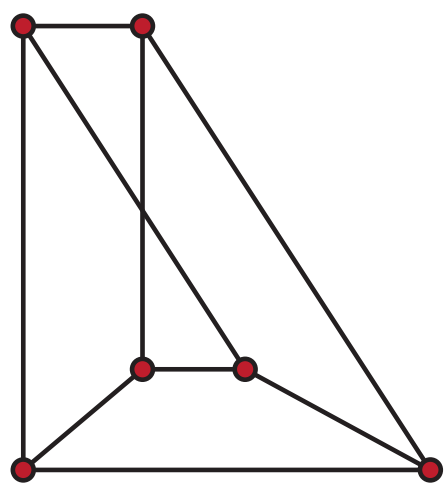

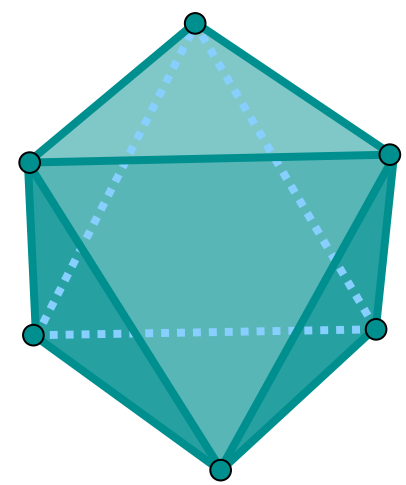

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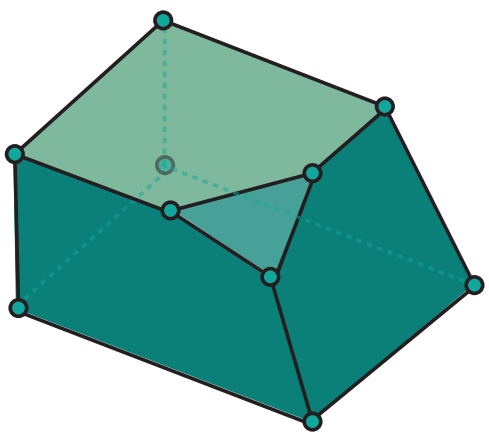

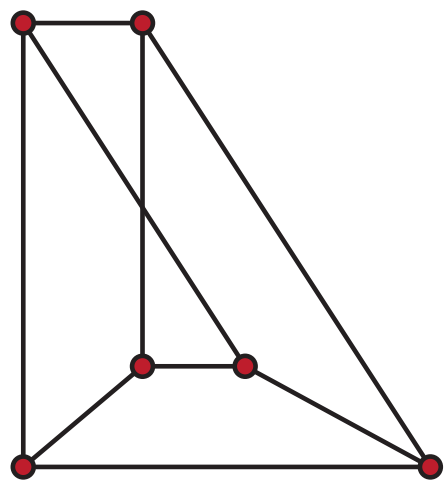

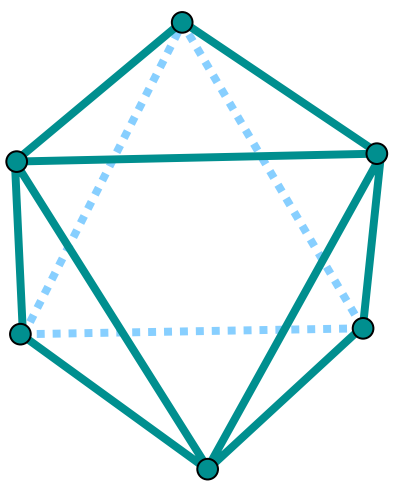

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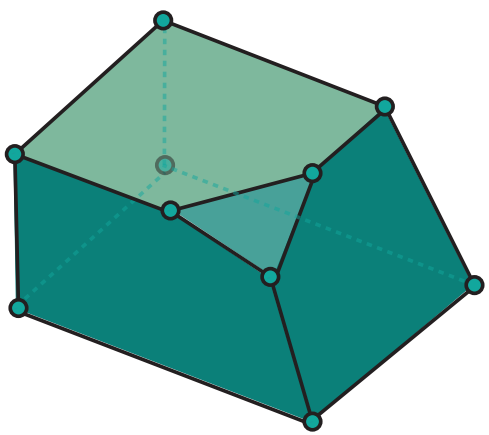

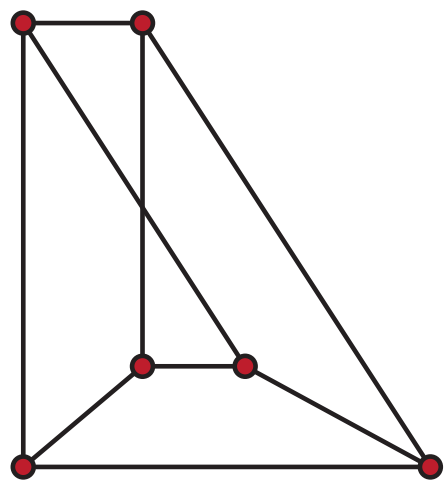

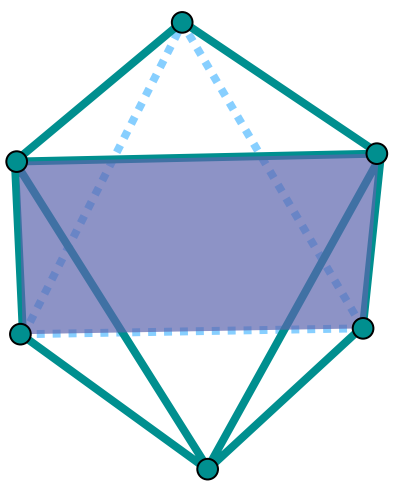

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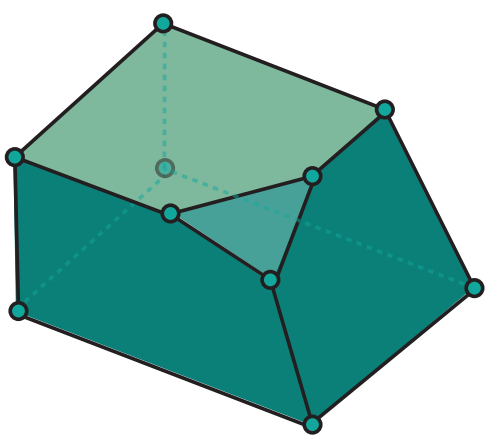

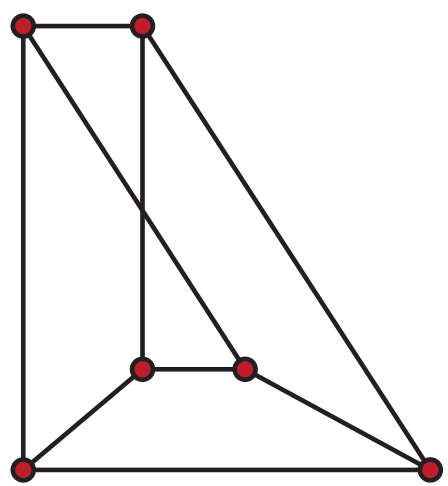

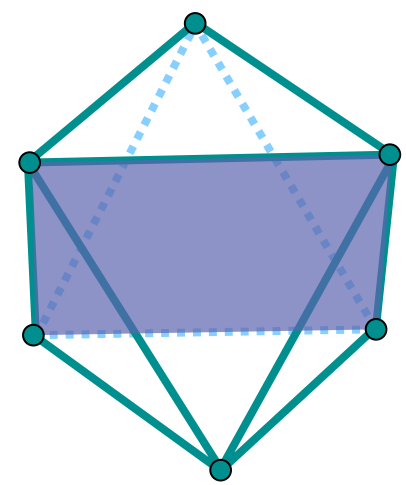

The moment polytope is a discrete representation of the orbit space M/T.

Chambers of the moment polytope

DEF: The chambers of the moment polytope are the open regions in

$$\Delta - \left\{ (d-1)\text{-dimensional facets} \right\}.$$

Questions:

- Can we count the chambers?
- Can we distinguish the chambers?

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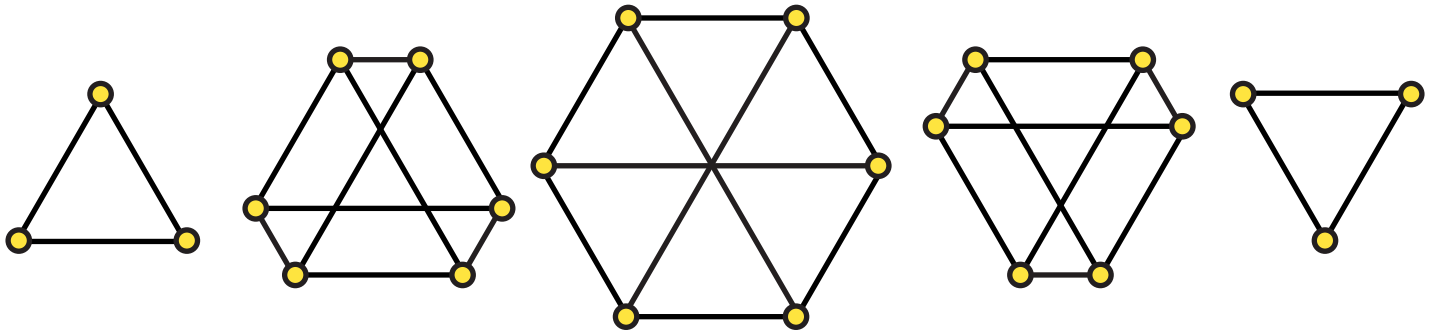

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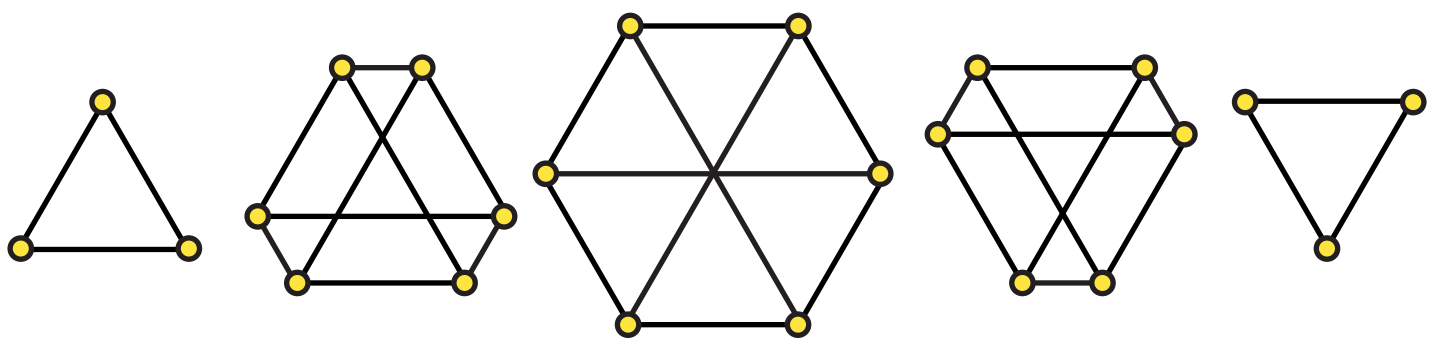

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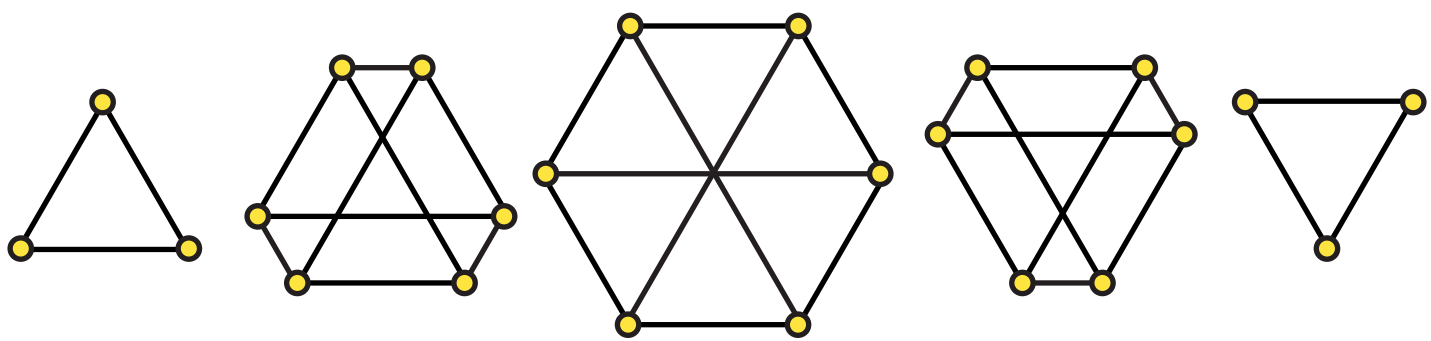


Billey, Guillemin and Rassart have shown that a generic permutahedron in $\mathbb{R}^3$ has 213; 229; 261; 277; 325; or 337 chambers!

Chambers of the moment polytope

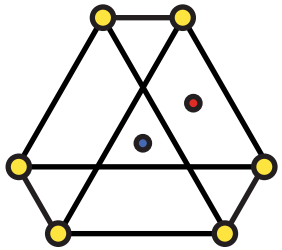
To any point μ in a chamber, we may associate an ideal

$$I_\mu \subseteq H_T^*(M).$$

<u>Theorem</u> [Goldin-H-Jeffrey]:
Suppose T acts on M Hamiltonianly with isolated fixed points, and μ_1 and μ_2 are elements of chambers of Δ. Then

$$I_{\mu_1} = I_{\mu_2}$$

if and only if μ_1 and μ_2 are in the same chamber of Δ.



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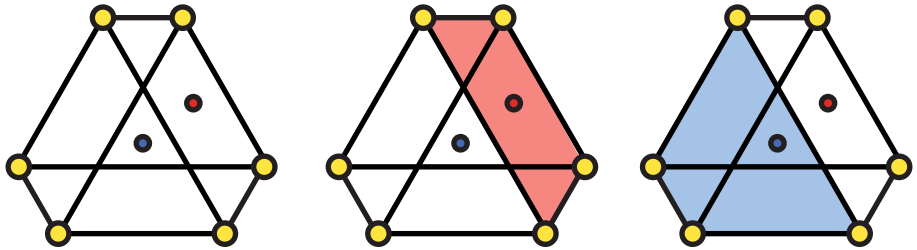

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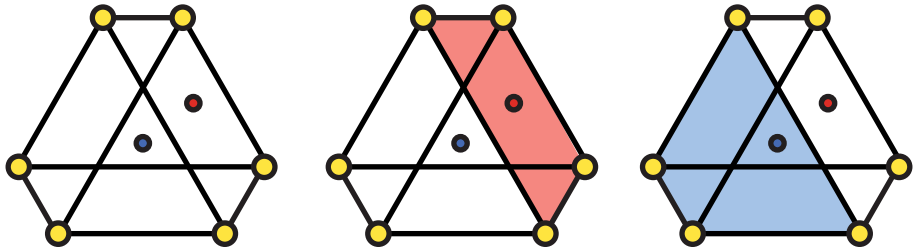

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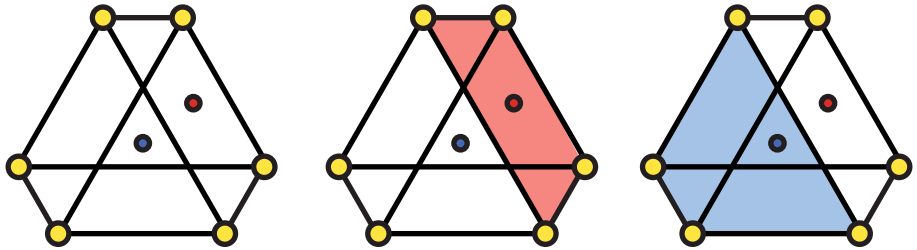

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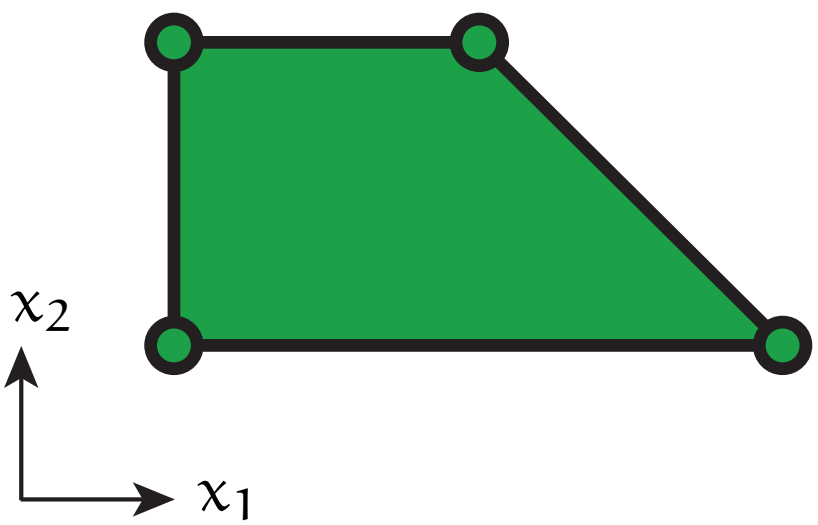

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Example

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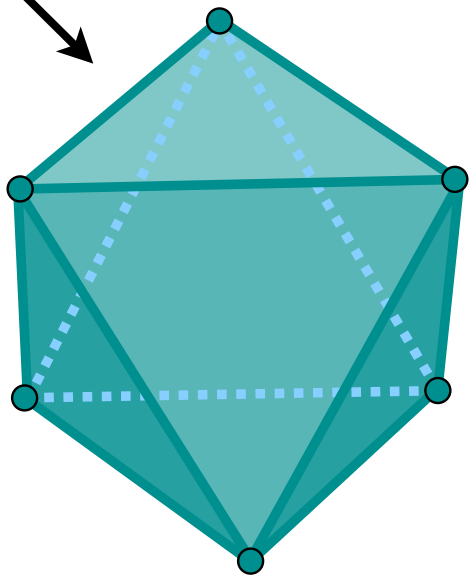

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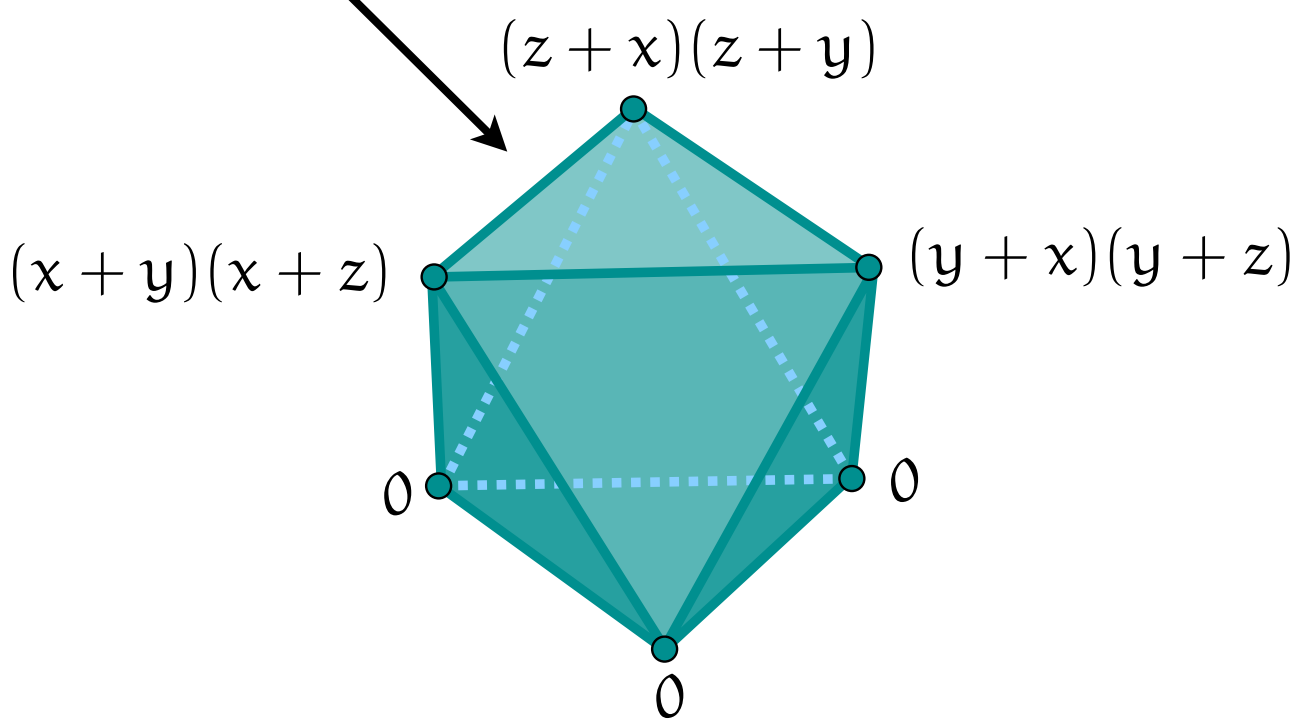

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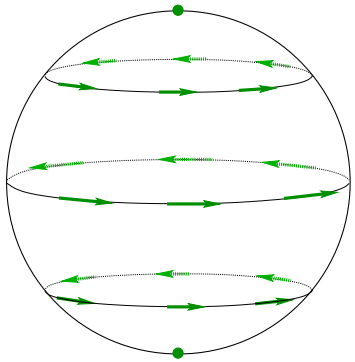

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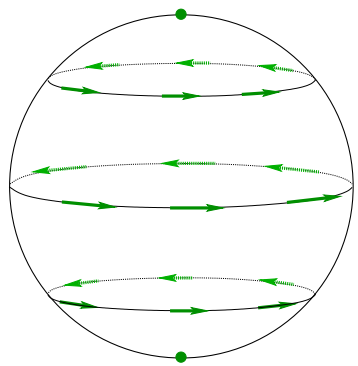

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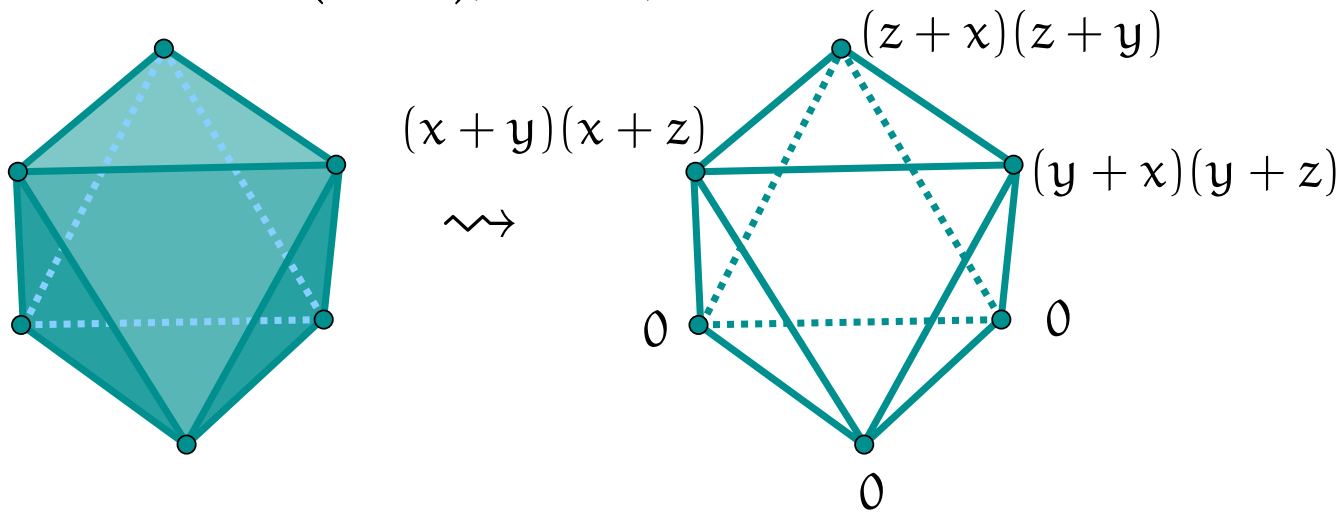

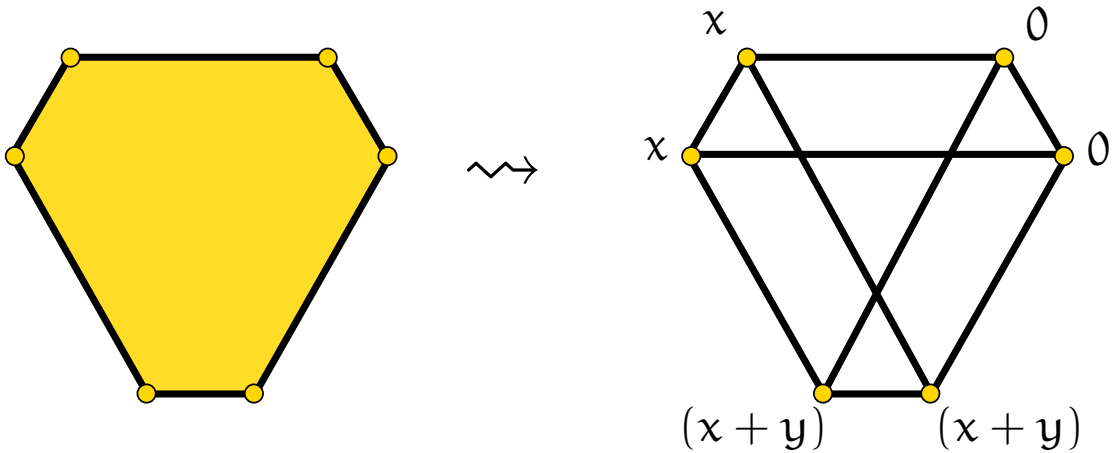

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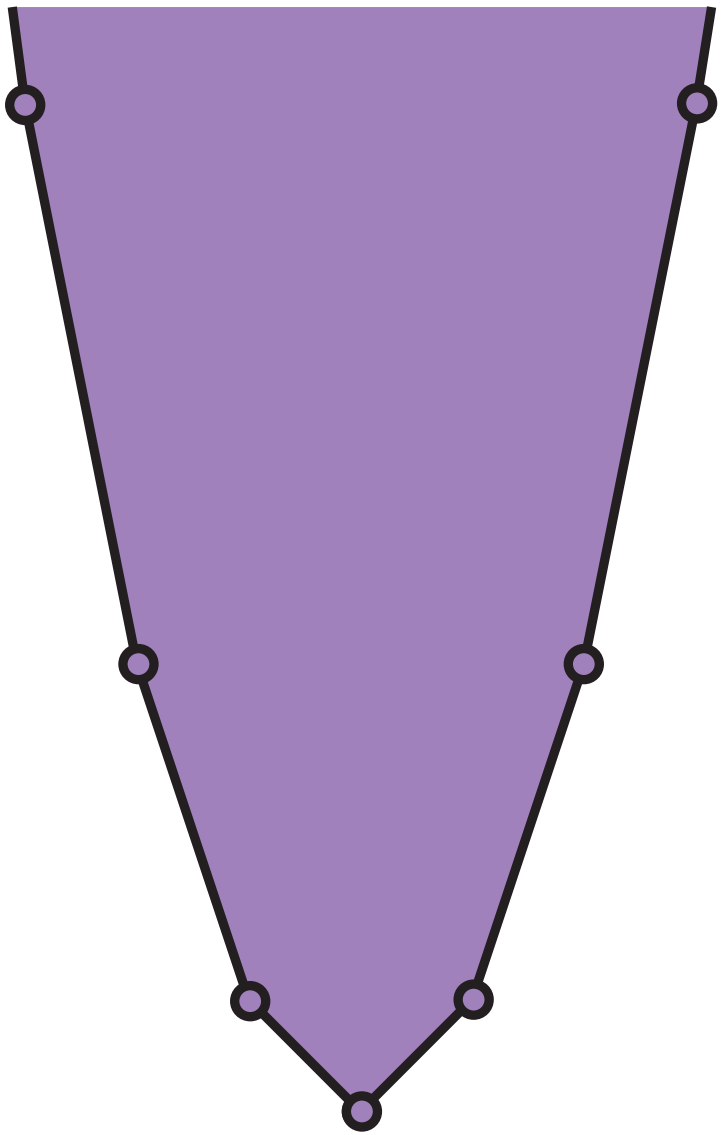

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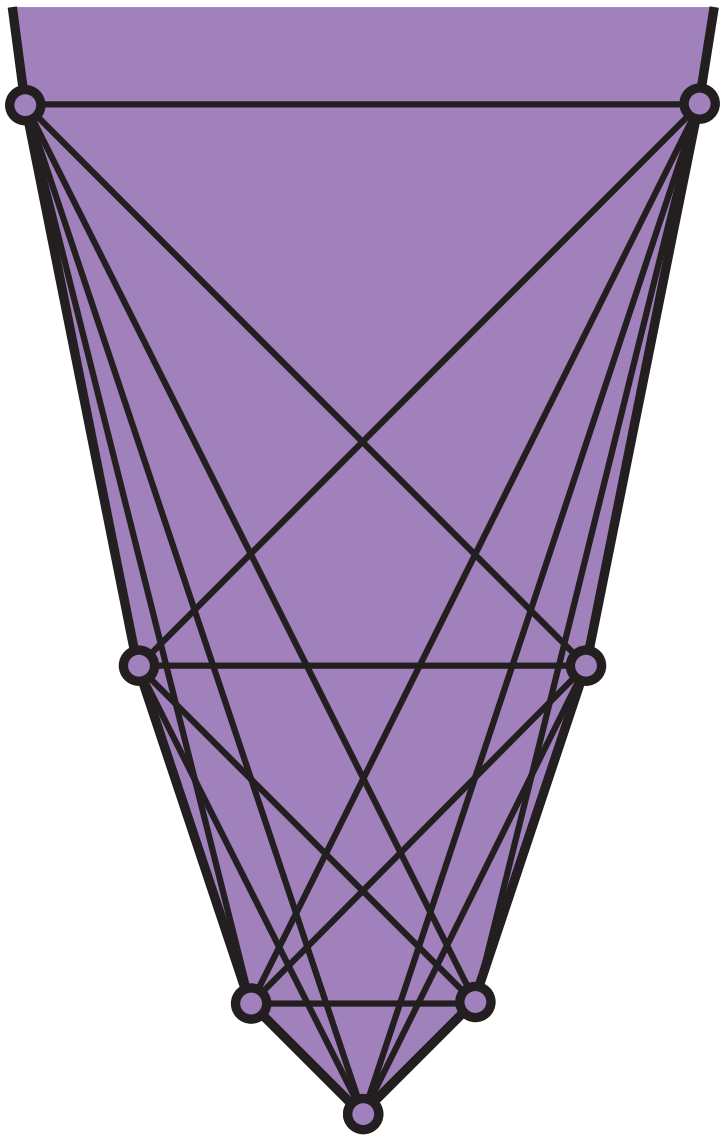

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To any point μ in a chamber, the submanifold

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Comments:

- The kernel is often combinatorially computable

  [Tolman-Weitsman; Goldin]
- The ring $H^*(M//T(\mu);\mathbb{R})$ does NOT distinguish the chambers!
- A similar result is known in K-theory

  [Harada-Landweber]
- Similar results can be formulated for stringy invariants

  [Goldin-H-Knutson; Goldin-Harada-H-Kimura]

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<u>Theorem</u> [Hausmann-Faber-Schütz]:

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Comments:

- The proof is algebraic, passing to the real points
- Does a similar result hold for $\mathcal{G}r(k, \mathbb{C}^d)$, for some values of d?
- How does this relate to these questions for $\mathcal{F}\ell ags(\mathbb{C}^d)$?