ENUMERATION OF POLYOMINOES INSCRIBED IN A RECTANGLE

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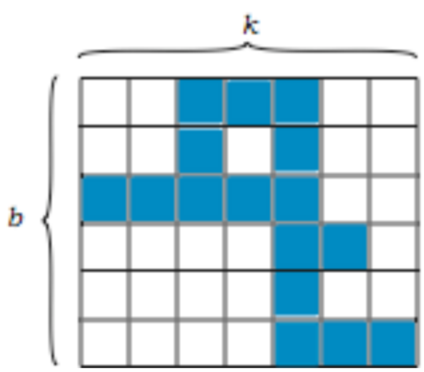
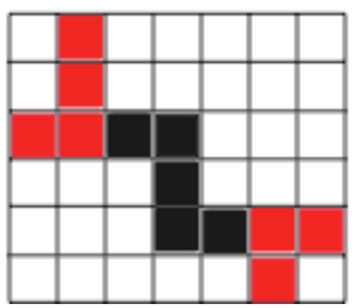
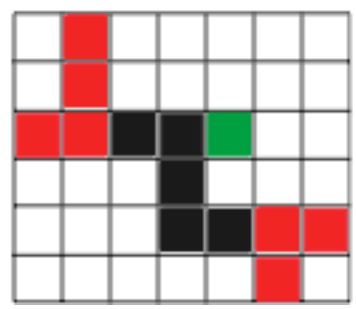


Polyomino inscribed in a rectangle

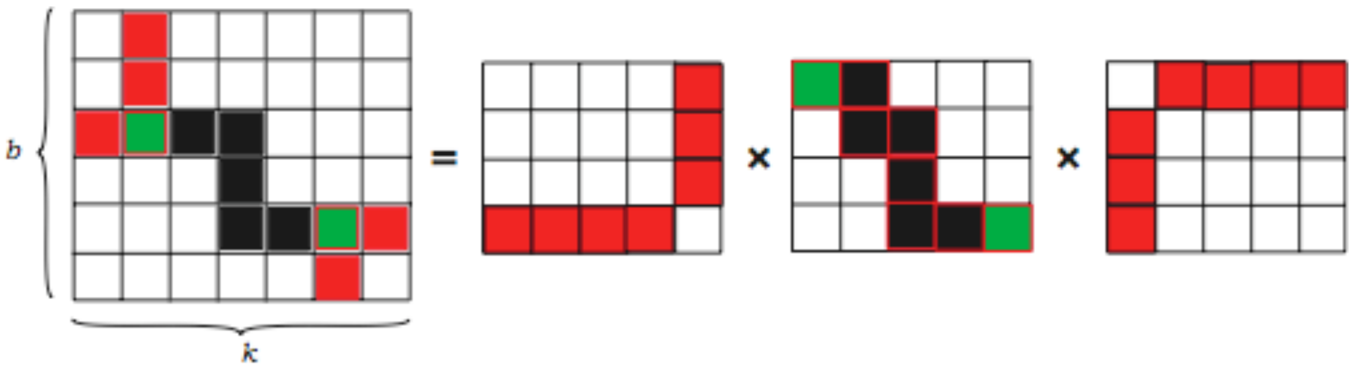

Polyomino with minimal area


Polyomino with minimal +1 area

Problem. Enumerate polyominoes inscribed in a rectangle !

- **Enumeration of polyominoes with minimal area**
  - area=$b+k$-1
  - they are disposed along a diagonal of the rectangle

First decomposition.



minimal = *hook* × *stair* × *hook*

Building blocs :

hook$(b,k)=1$ (the corner cell is fixed)

$$\Rightarrow \quad Hook(x,y)=1+\sum_{b,k\geq 1} x^b y^k = 1+\frac{xy}{(1-x)(1-y)}$$

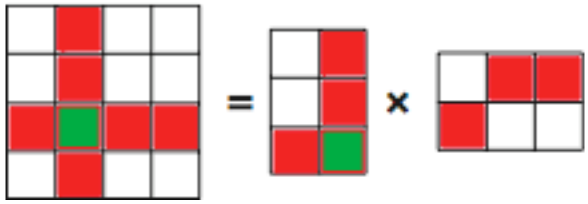
$$stair(b,k) = \binom{b+k-2}{b-1}$$

$$\Rightarrow \quad Stair(x,y)=\sum_{b,k\geq 1}\binom{b+k-2}{b-1}x^b y^k = \frac{xy}{(1-x-y)}$$

Polyominoes on one diagonal :

$$\Rightarrow \quad P_{min,\backslash}(x,y)=\left(1+\frac{xy}{(1-x)(1-y)}\right)^2 \frac{xy}{(1-x-y)}$$

Polyominoes on two diagonals: Crosses



cross = *hook* × (*hook* – *corner cell*)

except for polyominoes on one row or one column

$$\Rightarrow \qquad Cross(x,y)=\frac{xy}{(1-x)^2(1-y)^2}-\frac{xy^2}{(1-y)^2}-\frac{x^2y}{(1-x)^2}$$

Inclusion-exclusion.

$$P_{min}(x,y)= P_{min,\backslash}(x,y)+P_{min,/}(x,y)-\text{cross}(x,y)$$

$$\Rightarrow \qquad P_{min}(x,y)= \sum_{b,k\geq 1} p_{min}(b,k)x^b y^k$$

$$\left(1+\frac{xy}{(1-x)(1-y)}\right)^2\frac{2xy}{(1-x-y)}-\left(\frac{xy}{(1-x)^2(1-y)^2}-\frac{xy^2}{(1-y)^2}-\frac{x^2y}{(1-x)^2}\right)$$

Exact formulas :

$$P_{min}(b,k) = 8\binom{b+k-2}{b-1} - bk - 2(b-1)(k-1) - 6$$

if n = number of cells

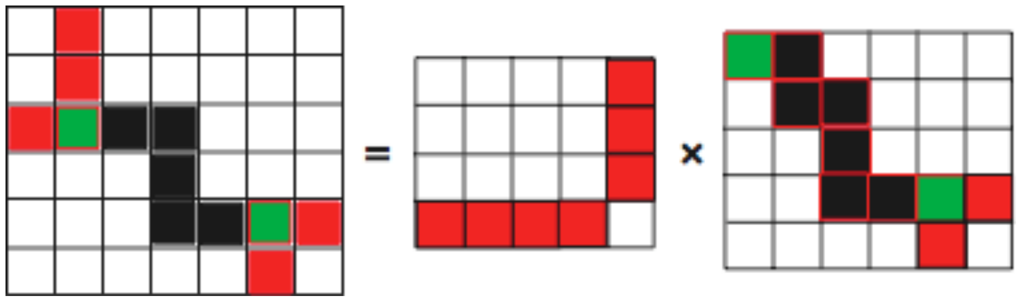
then $P_{min}(n)$ = number of polyominoes with n cells inscribed in any rectangle of perimeter $2n+2$

$$= \sum_{b=1}^{n} p_{min}(b, n-b+1)$$

$$= 2^{n+2} - \frac{1}{2}\left(n^3 - n^2 + 10n + 4\right)$$

$$P_{min}(z) = \sum_{b,k\geq 1} p_{min}(n) z^n = \frac{z^2(1-4z+8z^2-6z^3+4z^4)}{(1-z)^4(1-2z)}$$

Second decomposition :



minimal = hook × corner polyomino

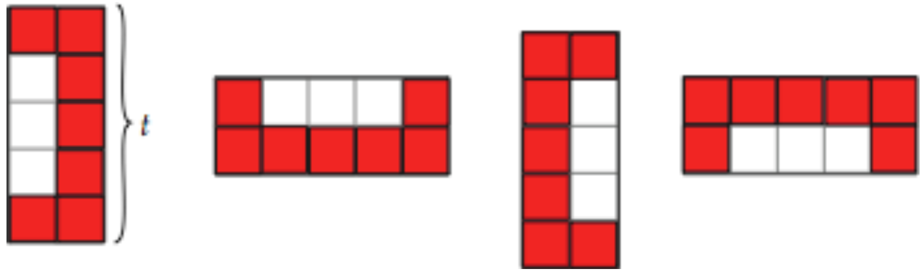
Corner polyominoes : inscribed polyominoes with min area and one cell in a given corner of the rectangle.

$$p_c(b,k) = \begin{cases} 1 & \text{if } b = 1 \text{ or } k = 1 \\ p_c(b-1,k) + p_c(b,k-1) + 1 & \text{otherwise} \end{cases}$$

$$= 2\binom{b+k-2}{b-1} - 1 \quad \text{for } b,k \geq 1$$

- **Polyominoes with *min+1* area**

Benches :



P is an inscribed polyomino of area *min*+1

⇔

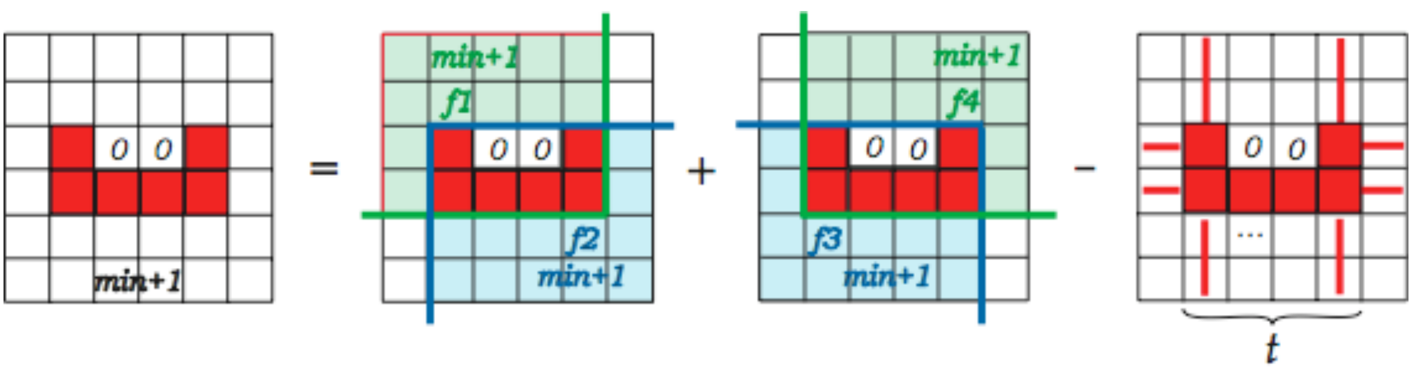
P contains exactly one bench

To construct all *min*+1 polyominoes that contain a given bench B:

1- Fix the position of the bench B in a $b \times k$ rectangle R.

2- Complete the bench into a polyomino with area *min+1* in two opposite regions; f_1--f_2 or f_3--f_4.

3- Use inclusion-exclusion and remove polyominoes that belong to both diagonals (i.e. hooks).

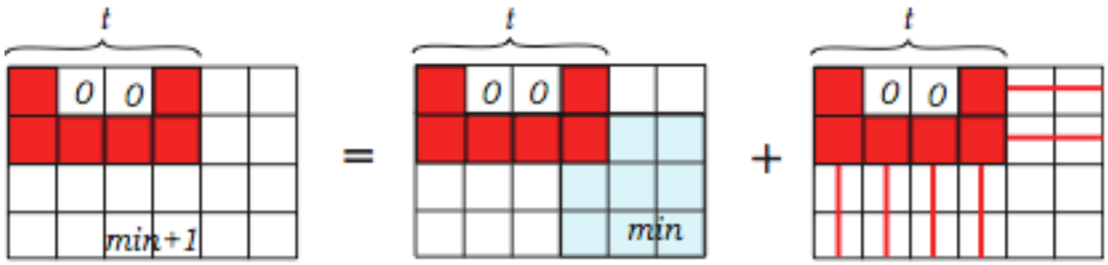


$P_{min+1}(B) \quad = \quad f_1 f_2 \quad + \quad f_3 f_4 \quad -8t$

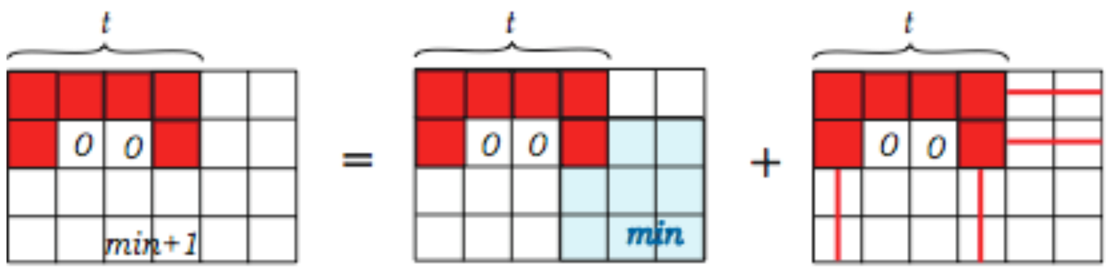
To obtain all inscribed *min*+1 polyominoes :

4- sum over all benches B in the rectangle R.

Case 1. The bench B is in a corner.



$$P_1(t,b,k) = \text{Corner polyomino} + \text{Hook},$$
$$= 2\binom{b+k-t-2}{b-2} + 2(t-1)$$

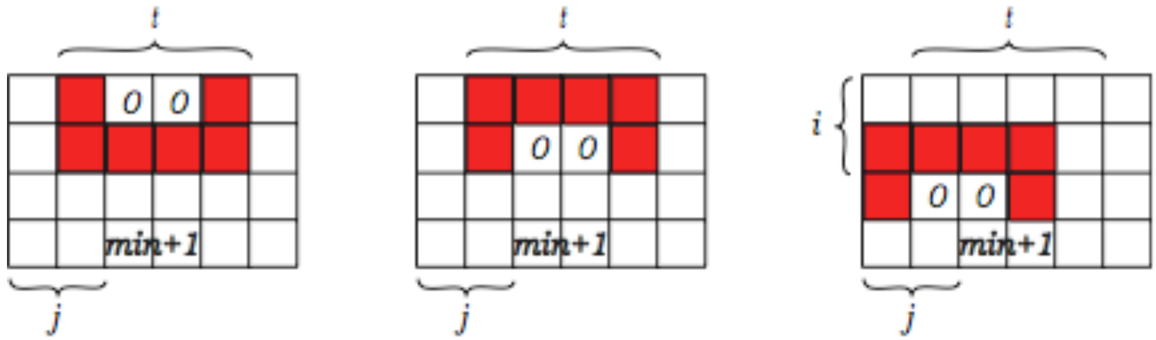


$$P_2(t,b,k) = \text{Corner polyomino} + \text{Hook},$$
$$= 2\binom{b+k-t-2}{b-2} + 2$$

Proposition. *The number $g_1(b,k)$ of polyominoes of area min+1 inscribed in a bxk rectangle with a bench in any corner of the rectangle is*

$$g_1(b,k) = \left(4\sum_{t=3}^{k-1} p_1(t,b,k) + 4\right) + \left(4\sum_{t=3}^{k-1} p_2(t,b,k) + 2k\right)$$
$$+ \left(4\sum_{t=3}^{b-1} p_1(t,b,k) + 4\right) + \left(4\sum_{t=3}^{b-1} p_2(t,b,k) + 2b\right)$$
$$= 16\left[\binom{b+k-4}{b-1} + \binom{b+k-4}{k-1}\right] + 2k(2k-1) + 2b(2b-1) - 72$$

Case 2. The bench is on one side of the rectangle and not in a corner.



Proposition. *The number $g_2(b,k)$ of polyominoes of area min+1 inscribed in a $b\times k$ rectangle with a bench touching exactly one side of the rectangle is*

$$g_2(b,k) = 32\left[\binom{b+k-4}{b} + \binom{b+k-4}{k}\right] + 8\left[10\binom{b+k-4}{b-2} + \binom{b+k-4}{b-1} + \binom{b+k-4}{k-1}\right] + \frac{4}{3}(b^3+k^3) - 28(b^2+k^2) - 48bk + \frac{164}{3}(b+k) + 4(bk^2+b^2k) + 144$$

Case 3. The bench touches no side of the rectangle.

Proposition. *The number $g_3(b,k)$ of polyominoes of area min+1 inscribed in a b×k rectangle with a bench touching no side of the rectangle is*

$$g_3(b,k) = \frac{8}{3}\Big[-12\Big[\binom{b+k-4}{b}+\binom{b+k-4}{k}\Big]+$$
$$6(b+k-6)\Big[\binom{b+k-4}{b-1}+\binom{b+k-4}{k-1}\Big]-60\binom{b+k-4}{b-2}$$
$$+18\binom{b+k-2}{b-1}-(b^3+k^3)+15(b^2+k^2)$$
$$-6(bk^2+b^2k)-48bk-56(b+k)+24\Big]$$

Case 4. 2×2 benches.

Proposition. *The number $p_{2\times2}(b,k)$ of polyominoes of area min+1 inscribed in a bxk rectangle with a bench touching no side of the rectangle is*

$$\begin{cases} 4(b+k-4) & \text{if } b=2, k\ge 3 \text{ or } k=2, b\ge 3 \\ 8\Big[\binom{b+k-4}{b-2}+2\binom{b+k-4}{b-1}+2\binom{b+k-4}{k-1}-3\Big] & \text{if } b=3 \text{ or } k=3 \\ 8\Big[\Big(\binom{b+k-4}{b-2}+1\Big)(b+k-2)-bk\Big] & \text{if } b,k\ge 4 \end{cases}$$

All cases.

Theorem. For *$b,k\geq 3$, the number $p_{\min+1}(b,k)$ of polyominoes of area min+1 inscribed in a b×k rectangle is*

$$p_{\min+1}(b,k) = g_1(b,k) + g_2(b,k) + g_3(b,k) + p_{2\times 2}(b,k)$$

$$= 8(b+k-22)\binom{b+k-4}{b-2} + \frac{8(2k^2+2kb+k-13k+13)}{(k-2)}\binom{b+k-4}{b-1}$$

$$+\frac{8(2b^2+2kb+b-13b+13)}{(b-2)}\binom{b+k-4}{k-1} + 48\binom{b+k-2}{b-1} - \frac{4}{3}(b^3+k^3)$$

$$-12(b^2k+bk^2)+16(b^2+k^2)+72bk-\frac{266}{3}(b+k)+120$$

Corollary. For integers *$n\geq 4$, the number $p_{\min+1}(n)$ of polyominoes of area n inscribed in any rectangle of perimeter 2n is given by*

$$p_{min+1}(n) = \sum_{b=2}^{n-2} p_{min+1}(b,n-b)$$

$$= 2^n\left(\frac{4}{5}+\frac{22}{5}n\right) - \frac{1}{3}\left(8n^4-88n^3+430n^2-902n+636\right)$$

● **Polyominoes with no loop** (lattice trees) **and *min*+1 area.**

Corollary. The number $\ell_{min+1}(b,k)$ of lattice trees inscribed in a $b \times k$ *rectangle with area min+1 is*

$$\ell_{min+1}(b,k) = f_{min+1}(b,k) - f_{2\times 2}(b,k)$$

Corollary. For integers $n \geq 5$, the number $\ell_{min+1}(n)$ of lattice trees *of area n inscribed in any rectangle of perimeter 2n is given by*

$$\ell_{min+1}(n) = \sum_{b=2}^{n-2} \ell_{min+1}(b, n-b)$$

$$= 2^{n+1}(n-1) - \frac{2}{3}\left(4n^4 - 46n^3 + 227n^2 - 473n + 318\right)$$

Next ...

● More recurrences, exact formulae, generating functions

● Minimal 3D polyominoes