Problem. Enumerate polyominoes inscribed in a rectangle!
Enumeration of polyominoes with minimal area

- area = b + k - 1
- they are disposed along a diagonal of the rectangle

**First decomposition.**

\[
\begin{align*}
\text{minimal} & = \text{hook} \times \text{stair} \times \text{hook} \\
\end{align*}
\]

**Building blocs:**

\[
\begin{align*}
\text{hook}(b,k) = 1 \quad \text{(the corner cell is fixed)} \\
\Rightarrow \text{Hook}(x,y) & = 1 + \sum_{b,k \geq 1} x^b y^k = 1 + \frac{xy}{(1 - x)(1 - y)} \\
\text{stair}(b,k) & = \binom{b + k - 2}{b - 1} \\
\Rightarrow \text{Stair}(x,y) & = \sum_{b,k \geq 1} \binom{b + k - 2}{b - 1} x^b y^k = \frac{xy}{(1 - x - y)} \\
\end{align*}
\]

Polyominoes on one diagonal:

\[
\begin{align*}
\Rightarrow \quad P_{\text{min,\text{\_}}}(x,y) & = \left(1 + \frac{xy}{(1 - x)(1 - y)}\right)^2 \frac{xy}{(1 - x - y)} \\
\end{align*}
\]
Polyominoes on two diagonals: Crosses

cross = hook × (hook – corner cell)

except for polyominoes on one row or one column

\[
\Rightarrow \quad \text{Cross}(x,y) = \frac{xy}{(1-x)^2(1-y)^2} - \frac{xy^2}{(1-y)^2} - \frac{x^2y}{(1-x)^2}
\]

Inclusion-exclusion.

\[
P_{\min}(x,y) = P_{\min,\setminus}(x,y) + P_{\min,/}(x,y) - \text{cross}(x,y)
\]

\[
\Rightarrow \quad P_{\min}(x,y) = \sum_{b,k \geq 1} p_{\min}(b,k) x^b y^k
\]

\[
\left(1 + \frac{xy}{(1-x)(1-y)}\right)^2 \frac{2xy}{(1-x-y)} - \left(\frac{xy}{(1-x)^2(1-y)^2} - \frac{xy^2}{(1-y)^2} - \frac{x^2y}{(1-x)^2}\right)
\]
Exact formulas:

\[ P_{\text{min}}(b,k) = 8 \binom{b+k-2}{b-1} - bk - 2(b-1)(k-1) - 6 \]

if \( n \) = number of cells

then \( P_{\text{min}}(n) \) = number of polyominoes with \( n \) cells inscribed in any rectangle of perimeter \( 2n+2 \)

\[ = \sum_{b=1}^{n} p_{\text{min}}(b,n-b+1) \]

\[ = 2^{n+2} - \frac{1}{2} \left( n^3 - n^2 + 10n + 4 \right) \]

\[ P_{\text{min}}(z) = \sum_{b,k \geq 1} p_{\text{min}}(n)z^n = \frac{z^2(1 - 4z + 8z^2 - 6z^3 + 4z^4)}{(1 - z)^4(1 - 2z)} \]
Second decomposition:

\[
\text{minimal} = \text{hook} \times \text{corner polyomino}
\]

Corner polyominoes: inscribed polyominoes with min area and one cell in a given corner of the rectangle.

\[
p_c(b,k) = \begin{cases} 
1 & \text{if } b = 1 \text{ or } k = 1 \\
 p_c(b-1,k) + p_c(b,k-1) + 1 & \text{otherwise}
\end{cases}
\]

\[
= 2 \binom{b+k-2}{b-1} - 1 \quad \text{for } b,k \geq 1
\]
Polyominoes with $min+1$ area

Benches:

$P$ is an inscribed polyomino of area $min+1$ \iff $P$ contains exactly one bench

To construct all $min+1$ polyominoes that contain a given bench $B$:

1. Fix the position of the bench $B$ in a $b \times k$ rectangle $R$.
2. Complete the bench into a polyomino with area $min+1$ in two opposite regions; $f_1--f_2$ or $f_3--f_4$.
3. Use inclusion-exclusion and remove polyominoes that belong to both diagonals (i.e. hooks).

$P_{min+1}(B) = f_1 f_2 + f_3 f_4 - 8t$

To obtain all inscribed $min+1$ polyominoes:

4. sum over all benches $B$ in the rectangle $R$. 

6
Case 1. The bench $B$ is in a corner.

\[ P_1(t,b,k) = \text{Corner polyomino} + \text{Hook}, \]
\[ = 2 \left( \frac{b + k - t - 2}{b - 2} \right) + 2(t - 1) \]

\[ P_2(t,b,k) = \text{Corner polyomino} + \text{Hook}, \]
\[ = 2 \left( \frac{b + k - t - 2}{b - 2} \right) + 2 \]

**Proposition.** The number $g_1(b,k)$ of polyominoes of area $\min+1$ inscribed in a $b \times k$ rectangle with a bench in any corner of the rectangle is

\[
g_1(b,k) = \left( 4 \sum_{t=3}^{k-1} p_1(t,b,k) + 4 \right) + \left( 4 \sum_{t=3}^{k-1} p_2(t,b,k) + 2k \right) \\
+ \left( 4 \sum_{t=3}^{b-1} p_1(t,b,k) + 4 \right) + \left( 4 \sum_{t=3}^{b-1} p_2(t,b,k) + 2b \right) \\
= 16 \left[ \left( \frac{b + k - 4}{b - 1} \right) + \left( \frac{b + k - 4}{k - 1} \right) \right] + 2k(2k - 1) + 2b(2b - 1) - 72
\]
Case 2. The bench is on one side of the rectangle and not in a corner.

**Proposition.** The number $g_2(b,k)$ of polyominoes of area $\min+1$ inscribed in a $b \times k$ rectangle with a bench touching exactly one side of the rectangle is

$$
g_2(b,k) = 32 \left[ \binom{b+k-4}{b} + \binom{b+k-4}{k} \right] + \\
8 \left[ 10 \binom{b+k-4}{b-2} + \binom{b+k-4}{b-1} + \binom{b+k-4}{k-1} \right] + \\
\frac{4}{3} (b^3 + k^3) - 28(b^2 + k^2) - 48bk + \frac{164}{3} (b+k) \\
+ 4(bk^2 + b^2k) + 144
$$
Case 3. The bench touches no side of the rectangle.

**Proposition.** The number $g_3(b,k)$ of polyominoes of area $\min+1$ inscribed in a $b\times k$ rectangle with a bench touching no side of the rectangle is

$$g_3(b,k) = \frac{8}{3} \left[ -12 \left( \begin{array}{c} b+k-4 \\ b \end{array} \right) + \left( \begin{array}{c} b+k-4 \\ k \end{array} \right) + 6 \left( \begin{array}{c} b+k-6 \\ b-1 \end{array} \right) \left( \begin{array}{c} b+k-4 \\ b-1 \end{array} \right) + \left( \begin{array}{c} b+k-4 \\ k-1 \end{array} \right) - 60 \left( \begin{array}{c} b+k-4 \\ b-2 \end{array} \right) + 18 \left( \begin{array}{c} b+k-2 \\ b-1 \end{array} \right) - (b^3 + k^3) + 15(b^2 + k^2) - 6(bk^2 + b^2k) - 48bk - 56(b+k) + 24 \right]$$

Case 4. 2x2 benches.

**Proposition.** The number $p_{2\times2}(b,k)$ of polyominoes of area $\min+1$ inscribed in a $b\times k$ rectangle with a bench touching no side of the rectangle is

$$p_{2\times2}(b,k) = \begin{cases} 
4(b+k-4) & \text{if } b=2, k \geq 3 \text{ or } k=2, b \geq 3 \\
8 \left( \begin{array}{c} b+k-4 \\ b-2 \end{array} \right) + 2 \left( \begin{array}{c} b+k-4 \\ b-1 \end{array} \right) + 2 \left( \begin{array}{c} b+k-4 \\ k-1 \end{array} \right) - 3 & \text{if } b=3 \text{ or } k=3 \\
8 \left( \begin{array}{c} b+k-4 \\ b-2 \end{array} \right) + 1 \left( b+k-2 \right) - bk & \text{if } b,k \geq 4
\end{cases}$$
All cases.

**Theorem.** For $b, k \geq 3$, the number $p_{\text{min}+1}(b,k)$ of polyominoes of area $\text{min}+1$ inscribed in a $b \times k$ rectangle is

$$p_{\text{min}+1}(b,k) = g_1(b,k) + g_2(b,k) + g_3(b,k) + p_{2 \times 2}(b,k)$$

$$= 8(b + k - 22) \binom{b + k - 4}{b - 2} + \frac{8(2k^2 + 2kb + k - 13k + 13)}{(k - 2)} \binom{b + k - 4}{b - 1}$$

$$+ \frac{8(2b^2 + 2kb + b - 13b + 13)}{(b - 2)} \binom{b + k - 4}{k - 1} + 48 \binom{b + k - 2}{b - 1} - \frac{4}{3}(b^3 + k^3)$$

$$- 12(b^2k + bk^2) + 16(b^2 + k^2) + 72bk - \frac{266}{3}(b + k) + 120$$

**Corollary.** For integers $n \geq 4$, the number $p_{\text{min}+1}(n)$ of polyominoes of area $n$ inscribed in any rectangle of perimeter $2n$ is given by

$$p_{\text{min}+1}(n) = \sum_{b=2}^{n-2} p_{\text{min}+1}(b, n-b)$$

$$= 2^n \left( \frac{4}{5} + \frac{22}{5}n \right) - \frac{1}{3} \left( 8n^4 - 88n^3 + 430n^2 - 902n + 636 \right)$$
Polyominoes with no loop (lattice trees) and $\min + 1$ area.

**Corollary.** The number $\ell_{\min+1}(b,k)$ of lattice trees inscribed in a $b \times k$ rectangle with area $\min + 1$ is

$$\ell_{\min+1}(b,k) = f_{\min+1}(b,k) - f_{2 \times 2}(b,k)$$

**Corollary.** For integers $n \geq 5$, the number $\ell_{\min+1}(n)$ of lattice trees of area $n$ inscribed in any rectangle of perimeter $2n$ is given by

$$\ell_{\min+1}(n) = \sum_{b=2}^{n-2} \ell_{\min+1}(b,n-b)$$

$$= 2^{n+1}(n-1) - \frac{2}{3} \left(4n^4 - 46n^3 + 227n^2 - 473n + 318\right)$$
Next ...

- More recurrences, exact formulae, generating functions
- Minimal 3D polyominoes