Hypergeometric series with algebro-geometric dressing

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Universidad de Buenos Aires

FPSAC 2010, 08/05/10

Based on joint work:

The structure of bivariate rational hypergeometric functions (with Eduardo Cattani and Fernando Rodríguez Villegas) arXiv:0907.0790, to appear: IMRN.

Bivariate hypergeometric D-modules (with Laura Matusevich and Timur Sadykov) Advances in Math., 2005.

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Aim and plan of the talk

- **Aim:** *Show two sample results on bivariate hypergeometric series/recurrences with inspiration/proof driven by algebraic geometry.*

1. *First problem: Solutions to hypergeometric recurrences in* $\mathbb{Z}^2$.
2. *Second problem: Characterize hypergeometric rational series in* 2 *variables.*
3. *Definitions/properties concerning* A*-hypergeometric systems and toric residues.*

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Solutions to hypergeometric recurrences

$$\mathbf{A}_n := \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}, \quad F(\alpha,\beta,\gamma;x) = \sum_{n\geq 0} \mathbf{A_n} x^n.$$

$(c)_n = c(c+1)\dots(c+n-1)$, $(1)_n = n!$, Pochammer symbol

Key equivalence

The coefficients A_n satisfy the following recurrence:

$$(1+n)(\gamma+n)A_{n+1} - (\alpha+n)(\beta+n)A_n = 0 \qquad (1)$$

(1) is *equivalent* to the fact that $F(\alpha,\beta,\gamma;x)$ satisfies Gauss differential equation (Kummer, Riemann):

$$[\Theta(\Theta+\gamma-1) - x(\Theta+\alpha)(\Theta+\beta)](F) = 0, \quad \Theta = x\frac{d}{dx}$$

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So: A_{n+1}/A_n is the *rational* function of n: $(\alpha+n)(\beta+n)/(1+n)(\gamma+n)$.

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$$\mathbf{A}_n := \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}, \quad \gamma \notin \mathbb{Z}_{<0}, \quad F(\alpha,\beta,\gamma;x) = \sum_{n\geq 0} \mathbf{A}_n x^n.$$

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If we define $A_n = 0$ for all $n \in \mathbb{Z}_{<0}$, the coefficients A_n satisfy the recurrence:

$$(1+n)(\gamma+n)A_{n+1} - (\alpha+n)(\beta+n)A_n = 0, \quad \text{for all } n \in \mathbb{Z} \qquad (2)$$

(2) is *equivalent* to the fact that $F(\alpha,\beta,\gamma;x)$ satisfies Gauss differential equation:

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Solutions to hypergeometric recurrences

$$\mathbf{B}_n := \frac{(\alpha)_n(\beta)_n}{(\gamma)_n(\delta)_n}, \quad \gamma, \delta \notin \mathbb{Z}_{<0}, \quad G(\alpha,\beta,\gamma,\delta;x) = \sum_{n\geq 0} \mathbf{B}_n x^n.$$

Caveat

$$(\delta+n)(\gamma+n)B_{n+1} - (\alpha+n)(\beta+n)B_n = 0, \quad \text{for all } n \in \mathbb{N}. \tag{3}$$

but $G(\alpha,\beta,\gamma;x)$ does not satisfy the differential equation:

$$[(\Theta+\delta-1)(\Theta+\gamma-1) - x(\Theta+\alpha)(\Theta+\beta)](G) = 0.$$

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The normalization hides the initial condition

If we define $B_n = 0$ for all $n \in \mathbb{Z}_{<0}$, then

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Hypergeometric recurrences in two variables

Naive generalization

Let a_{mn}, $m, n \in \mathbb{N}$ such that there exist two rational functions $R_1(m, n)$, $R_2(m, n)$ expressible as *products of (affine) linear functions* in (m, n), such that

$$\frac{a_{m+1,n}}{a_{mn}} = R_1(m, n), \quad \frac{a_{m,n+1}}{a_{mn}} = R_2(m, n) \tag{5}$$

(with obvious *compatibility* conditions).

Write

$$R_1(m, n) = \frac{P_1(m, n)}{Q_1(m+1, n)}, \quad R_2(m, n) = \frac{P_2(m, n)}{Q_2(m, n+1)}.$$

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Hypergeometric recurrences in two variables

Naive generalization, suite

Consider the generating function $F(x_1, x_2) = \sum_{m,n\in\mathbb{N}} a_{mn}\, x_1^m x_2^n$ and the differential operators $(\theta_i = x_i \frac{\partial}{\partial x_i})$:

$$\Delta_1 = Q_1(\theta_1, \theta_2) - x_1 P_1(\theta_1, \theta_2) \quad \Delta_2 = Q_2(\theta_1, \theta_2) - x_2 P_2(\theta_1, \theta_2).$$

Then, the recurrences (5) in the coefficients a_{mn} are equivalent to $\Delta_1(F) = \Delta_2(F) = 0$ if $Q_1(0, n) = Q_2(m, 0) = 0$ and in this case, if we extend the definition of a_{mn} by 0, the recurrences

$$Q_1(m+1, n)a_{m+1,n} - P_1(m, n) = Q_2(m, n+1)a_{m,n+1} - P_2(m, n) = 0$$

hold for all $(m, n) \in \mathbb{Z}^2$.

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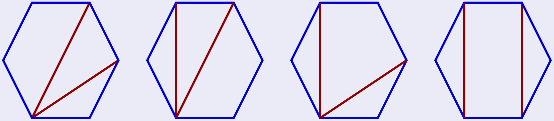
Two examples from combinatorics

Dissections

A subdivision of a regular n-gon into $(m+1)$ cells by means of nonintersecting diagonals is called a *dissection*.



How many dissections are there?

$$d_{m,n} = \frac{1}{m+1}\binom{n-3}{m}\binom{m+n-1}{m}; \quad 0 \le m \le n-3.$$

So, the generating function is naturally defined for (m,n) belonging to the lattice points in the *rational cone* $\{(a,b)/0 \le a \le b-3\}$ (and 0 outside).

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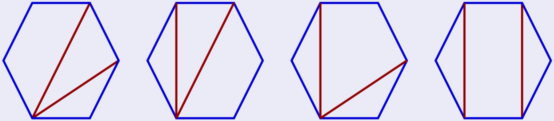

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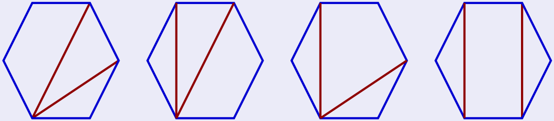

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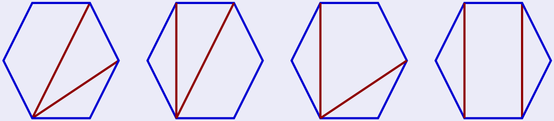

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[Example 9.2, Gessell and Xin, *The generating function of ternary trees and continued fractions*, EJC '06]

$$GX(x,y) = \frac{1-xy}{1-xy^2-3xy-x^2y} = \sum_{m,n\geq 0} \binom{m+n}{2m-n} x^m y^n,$$

where $\binom{a}{b}$ is defined as 0 if $b < 0$ or $a - b < 0$.

So we are summing over the lattice points in the convex rational cone $\{(a,b) \in \mathbb{R}^2 : 2a - b \geq 0,\ 2b - a \geq 0\} = \mathbb{R}_{\geq 0}(1,2) + \mathbb{R}_{\geq 0}(2,1)$. Or: the terms are defined over $\mathbb{Z}^2$ extending by 0 outside the cone.

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Two examples from combinatorics

[Example 9.2, Gessell and Xin, *The generating function of ternary trees and continued fractions*, EJC '06]

$$GX(x,y) = \frac{1-xy}{1-xy^2-3xy-x^2y} = \sum_{m,n\geq 0} \binom{m+n}{2m-n} x^m y^n,$$

where $\binom{a}{b}$ is defined as 0 if $b<0$ or $a-b<0$.
So we are summing over the lattice points in the convex rational cone $\{(a,b)\in\mathbb{R}^2 : 2a-b\geq 0,\ 2b-a\geq 0\} = \mathbb{R}_{\geq 0}(1,2)+\mathbb{R}_{\geq 0}(2,1)$. Or: the terms are defined over $\mathbb{Z}^2$ extending by 0 outside the cone.

Our results through an example

Data

Consider the hypergeometric terms $a_{m,n} = (-1)^n \frac{(2m-n+2)!}{n!\,m!\,(m-2n)!}$ for (m, n) integers with $m - 2n \geq 0, n \geq 0$, which satisfy the recurrences:

$$\frac{a_{m+1,n}}{a_{m,n}} = \frac{(2m-n+4)\,(2m-n+3)}{(m+1)\,(m+1-2n)} = \frac{\mathbf{P_1(m,n)}}{\mathbf{Q_1(m+1,n)}}$$

$$P_1(m,n) = (2m-n+4)\,(2m-n+3), \quad Q_1(m,n) = m\,(m-2n)$$

$$\frac{a_{m,n+1}}{a_{m,n}} = -\frac{(m-2n)\,(m-2n-1)}{(2m-n+2)\,(n+1)} = \frac{\mathbf{P_2(m,n)}}{\mathbf{Q_2(m,n+1)}}$$

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We have that the terms $t_{m,n} = a_{mn}$ for $m - 2n \geq 0, n \geq 0$ and $t_{(m,n)} = 0$ for any other $(m,n) \in \mathbb{Z}^2$, satisfy the recurrences:

$$Q_1(m+1,n)t_{m+1,n} - P_1(m,n)t_{m,n} = Q_2(m,n+1)t_{(m,n+1)} - P_2(m,n)t_{m,n} = 0. \quad (6)$$

Question

Which other terms $t_{m,n}$, $(m,n) \in \mathbb{Z}^2$ satisfy (6)?

Remark

When the linear forms in the polynomials P_i, Q_i defining the recurrences have generic constant terms, the solution is given by the Ore-Sato coefficients.

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There are four solutions $a_{mn}, b_{mn}, c_{mn}, d_{mn}$ (up to linear combinations), with generating series $F_1, \ldots, F_4$:

$$a_{m,n} = (-1)^n \frac{(2m-n+2)!}{n!\,m!\,(m-2n)!}, \quad F_1 = \sum_{\substack{m-2n\geq 0\\ n\geq 0}} a_{m,n}\, x_1^m\, x_2^n,$$

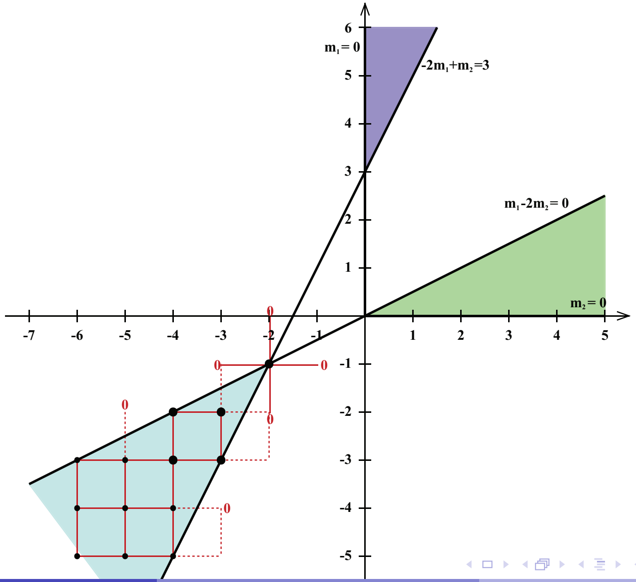
$$b_{m,n} = (-1)^m \frac{(2m-n-1)!}{n!\,m!\,(-2m+n+3)!}, \quad F_2 = \sum_{\substack{-2m+n\geq 3\\ m\geq 0}} b_{m,n}\, x_1^m\, x_2^n$$

$$c_{m,n} = (-1)^{m+n} \frac{(-m-1)!\,(-n-1)!}{(m-2n)!\,(-2m+n-3)!}, \quad F_3 = \sum_{\substack{m-2n\geq 0\\ -2m+n\geq 3}} c_{m,n}\, x_1^m\, x_2^n$$

$$d_{-2,-1} = 1, \quad F_4 = x_1^{-2}\, x_2^{-1}.$$

In all cases, $t_{mn} = 0$ outside the support of the series.

Pictorially



Explanations

- The generating functions F_i satisfy the *differential* equations:
  $[\Theta_1(\Theta_1 - 2\Theta_2) - x_1(2\Theta_1 - \Theta_2 + 4)(2\Theta_1 - \Theta_2 + 3)](F) = 0,$
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- Consider the system of *binomial* equations:

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  in the commutative polynomial ring $\mathbb{C}[\partial_1, \ldots, \partial_4]$.
- The zero set $q_1 = q_2 = 0$ has two irreducible components, one of degree 3 and mutiplicity 1, which intersects $(\mathbb{C}^*)^4$ (it is the twisted cubic), and another component "at infinity": $\{\partial_2 = \partial_3 = 0\}$, of degree 1 and multiplicity $1 = \min\{2 \times 2, 1 \times 1\}$.

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- The **multiplicity** of this only (*non homogeneous*) component at infinity is equal to the **dimension** of the space of solutions of the recurrences with finite support.

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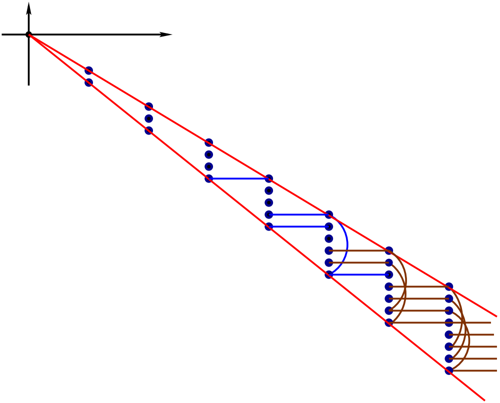
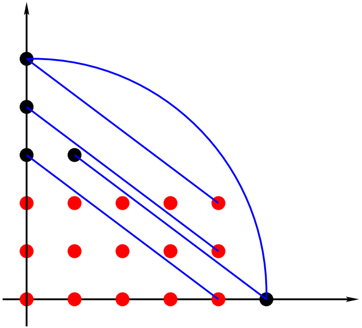
Finite recurrences and polynomial solutions





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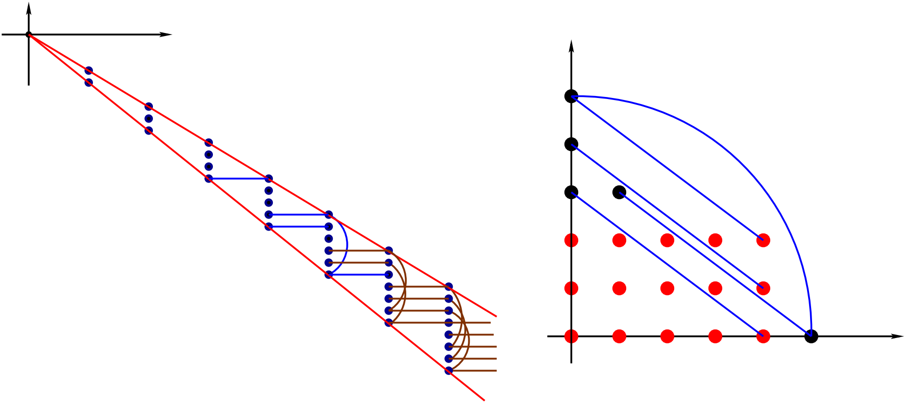

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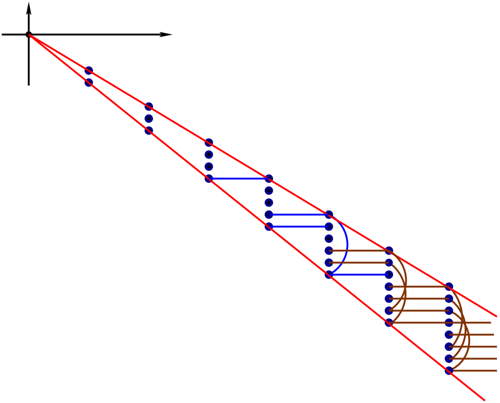

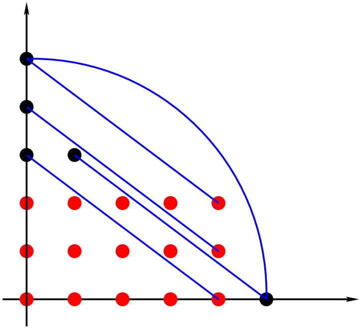

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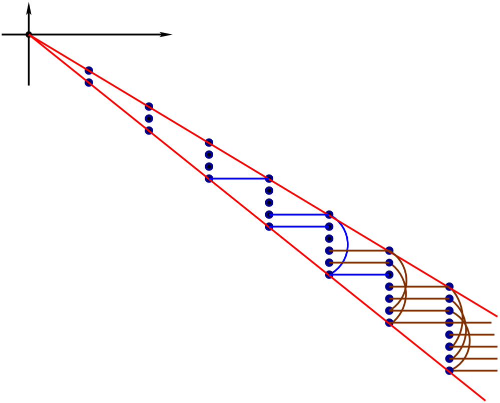

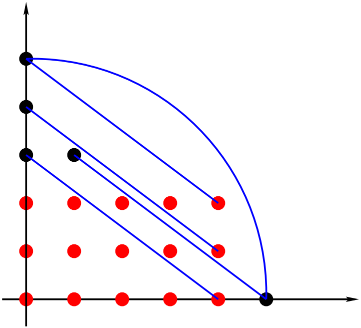

General picture

Let $B \in \mathbb{Z}^{n\times 2}$ with *rows* $b_1, \ldots, b_n$ satisfying $b_1 + \cdots + b_n = 0$.

$$P_i = \prod_{b_{ji}<0} \prod_{l=0}^{|b_{ji}|-1} (b_j \cdot \theta + c_j - l), \quad (7)$$

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where $b_j \cdot \theta = \sum_{k=1}^{2} b_{jk}\theta_{x_k}$.

The operators H_i are called *Horn operators* and generate the left ideal $\mathrm{Horn}(\mathcal{B}, c)$ in the Weyl algebra D_2. Call $d_i = \sum_{b_{ij}>0} b_{ij} = -\sum_{b_{ij}<0} b_{ij}$ the order of the operator H_i.

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Let $B \in \mathbb{Z}^{n\times 2}$ as above and let $A \in \mathbb{Z}^{(n-2)\times n}$ such that the columns $b^{(1)}, b^{(2)}$ of B span $\ker_{\mathbb{Q}}(A)$.

Write any vector $u \in \mathbb{R}^n$ as $u = u_+ - u_-$, where $(u_+)_i = \max(u_i, 0)$, and $(u_-)_i = -\min(u_i, 0)$.

Definition

$$T_i = \partial^{b_+^{(i)}} - \partial^{b_-^{(i)}}, \quad i = 1, 2.$$

The left D_n-ideal $H_B(c)$ is defined by:

$$H_B(c) = \langle T_1, T_2 \rangle + \langle A\cdot\theta - A\cdot c \rangle \subseteq D_n.$$

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Let $B \in \mathbb{Z}^{n\times 2}$ as above and let $A \in \mathbb{Z}^{(n-2)\times n}$ such that the columns $b^{(1)}, b^{(2)}$ of B span $\ker_{\mathbb{Q}}(A)$.
Write any vector $u \in \mathbb{R}^n$ as $u = u_+ - u_-$, where $(u_+)_i = \max(u_i, 0)$, and $(u_-)_i = -\min(u_i, 0)$.

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Theorem

[D.- Matusevich - Sadykov '05] For generic complex parameters $c_1, \ldots, c_n$, the ideals $\text{Horn}\,(\mathcal{B}, c)$ and $H_{\mathcal{B}}(c)$ are holonomic. Moreover,

$$\text{rank}(H_{\mathcal{B}}(c)) = \text{rank}(\text{Horn}\,(\mathcal{B}, c)) = d_1 d_2 - \sum_{\substack{b_i,\, b_j \\ \textit{depdt}}} \nu_{ij} = g \cdot \text{vol}(A) + \sum_{\substack{b_i,\, b_j \\ \textit{indepdt}}} \nu_{ij}\ ,$$

where the the pairs b_i, b_j of rows lie in opposite open quadrants of $\mathbb{Z}^2$.

Remarks

Solutions to recurrences with finite support correspond to (Laurent) polynomial solutions. These solutions come from (non homogeneous) primary components at infinity of the binomial ideal $\langle T_1, T_2 \rangle$. There are $\sum \nu_{ij}$ many linearly independent. For special parameters a special study is needed, along the lines in [D. - Matusevich and Miller '10].

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Moral of this story

Key to the answer it the homogenization and translation to the A-side!

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Examples of rational bivariate hypergeometric series

The proof in the talk!

Lemma: The series $f_{(s_1,s_2)}(x) := \sum_{m\in\mathbb{N}^2} \frac{(s_1m_1+s_2m_2)!}{(s_1m_1)!(s_2m_2)!} x_1^{m_1}x_2^{m_2}$.

is a rational function for all $(s_1, s_2) \in \mathbb{N}^2$.

Proof: $f_{(0,0)}(x_1, x_2) = \sum_{m\in\mathbb{N}^2} x_1^{m_1}x_2^{m_2} = \frac{1}{(1-x_1)(1-x_2)}$,

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Rational bivariate hypergeometric series

Question

When is a hypergeometric series in 2 variables rational?

Let $c^i = (c^i_1, c^i_2)$ and $d^j = (d^j_1, d^j_2)$ for $i = 1, \ldots, r; j = 1, \ldots, s$ be vectors in $\mathbb{N}^2$. When is the series

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Gessell and Xin´s example of a rational bivariate hypergeometric series

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What if the cone is not the first orthant?

We had

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where we are summing over the lattice points in the (pointed) non unimodular convex cone $\mathbb{R}_{\geq 0}(1,2) + \mathbb{R}_{\geq 0}(2,1)$.

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Gessell and Xin´s example of a rational bivariate hypergeometric series

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The general result

Data

Suppose we are given linear functionals

$$\ell_i(m_1, m_2) := \langle b_i, (m_1, m_2)\rangle + k_i\,, \quad i = 1, \ldots, n,$$

where $b_i \in \mathbb{Z}^2 \backslash \{0\}$, $k_i \in \mathbb{Z}$ and $\sum_{i=1}^n b_i = 0$.

Take $\mathcal{C}$ a rational convex cone. The bivariate series:

$$\phi(\mathbf{x_1}, \mathbf{x_2}) = \sum_{m \in \mathcal{C} \cap \mathbb{Z}^2} \frac{\prod_{\ell_i(m)<0} (-1)^{\ell_i(m)} (-\ell_i(m) - 1)!}{\prod_{\ell_j(m)>0} \ell_j(m)!} x_1^{m_1} x_2^{m_2}. \qquad (10)$$

is called a *Horn series*.

The coefficients c_m of ϕ satisfy hypergeometric recurrences: for $j = 1, 2$, and any $m \in \mathcal{C} \cap \mathbb{Z}^2$ such that $m + e_j$ **also** lies in $\mathcal{C}$:

$$\frac{c_{m+e_j}}{c_m} = \frac{\prod_{b_{ij}<0} \prod_{l=0}^{-b_{ij}+1} \ell_i(m) - l}{\prod_{b_{ij}>0} \prod_{l=1}^{b_{ij}} \ell_i(m) + l}.$$

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Theorem

[Cattani, D.-, R. Villegas '09]

If the Horn series $\phi(\mathbf{x_1}, \mathbf{x_2})$ is a rational function then: either

(i) $n = 2r$ *is even and, after reordering we may assume:*

$$b_1 + b_{r+1} = \cdots = b_r + b_{2r} = 0, \text{ or} \tag{11}$$

(ii) B *consists of* $n = 2r + 3$ *vectors and, after reordering, we may assume that* $b_1, \ldots, b_{2r}$ *satisfy* (11) *and* $b_{2r+1} = s_1\nu_1$, $b_{2r+2} = s_2\nu_2$, $b_{2r+3} = -b_{2r+1} - b_{2r+2}$, *where* ν_1, ν_2 *are the primitive, integral, inward-pointing normals of* $\mathcal{C}$ *and* s_1, s_2 *are positive integers.*

Moreover, ϕ *can be expressed as a residue.*

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$\phi(x) = GX(-x) = \sum_{m \in \mathcal{C}\cap\mathbb{Z}^2} (-1)^{m_1+m_2} \binom{m_1+m_2}{2m_1-m_2} x_1^{m_1} x_2^{m_2}$ is a *Horn* series.

We read the lattice vectors $b_1 = (-1,-1)$, $b_2 = (-1,2)$, $b_3 = (2,-1)$, and we enlarge them to a configuration B by adding the vectors $b_4 = (1,0)$ and $b_5 = (-1,0)$.

B is the *Gale dual* of the configuration A:

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 2 & 0 & 3 \end{pmatrix}$$

and $\phi(x)$ is the dehomogenization of a toric residue associated to $f_1 = z_1 + z_2 t + z_3 t^2, f_2 = z_4 + z_5 t^3$.

In inhomogeneous coordinates we have the not so nice expression:

$$\phi(x) = \sum_{\eta^3 = -x_2/x_1} \mathrm{Res}_\eta \left(\frac{x_2 t/(x_2 + x_2 t - t^2)}{x_2 + x_1 t^3}\, dt \right),$$

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Outline of the proof

A key lemma about Laurent expansions of rational functions $+$ a nice ingredient: the *diagonals* of a rational bivariate power series define classical hypergeometric algebraic univariate functions. [Polya '22, Furstenberg '67, Safonov '00].

Number theoretic + monodromy ingredients: we use Theorem M below to reduce to the algebraic hyperg. functions classified by [Beukers-Heckmann '89], [F. R. Villegas '03, Bober '08]

Many previous results on A-hypergeometric functions, allow us to analyze the possible Laurent expansions of rational hypergeometric solutions and to construct **rational** solutions using toric residues. [Saito-Sturmfels-Takayama ´99; Cattani, D.-Sturmfels '01, 02; Cattani - D. ´04] .

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Diagonals of bivariate series

Given a bivariate power series

$$f(x_1, x_2) \;:=\; \sum_{n,m \geq 0} a_{m,n} x_1^m x_2^n \qquad (12)$$

and $\delta = (\delta_1, \delta_2) \in \mathbb{Z}^2_{>0}$, with $\gcd(\delta_1, \delta_2) = 1$, we define the δ-diagonal of f as:

$$f_\delta(t) \;:=\; \sum_{r \geq 0} a_{\delta_1 r, \delta_2 r}\, t^r \,. \qquad (13)$$

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Laurent series of rational functions

Let $p, q \in \mathbb{C}[x_1, x_2]$ **coprime**, $f = p/q$, $N(q) \subset \mathbb{R}^2$ the Newton polytope of q, v_0 be a vertex of $N(q)$, v_1, v_2 the adjacent vertices, indexed counterclockwise.

Hence, $N(q) \subset v_0 + \mathbb{R}_{>0} \cdot (v_1 - v_0) + \mathbb{R}_{>0} \cdot (v_2 - v_0)$.

So, $f(x)$ has a convergent Laurent series expansion with support contained in $x^w + C$ for suitable $w \in \mathbb{Z}^2$ [GKZ], where C is the cone

$$C = \mathbb{R}_{\geq 0}\,(v_1 - v_0) + \mathbb{R}_{\geq 0}\,(v_2 - v_0).$$

Key Lemma

*The support of the series is **not** contained in any subcone of the form $x^{w'} + C'$, with C' is **properly contained** in C.*

Laurent series of rational functions

Let $p, q \in \mathbb{C}[x_1, x_2]$ **coprime**, $f = p/q$, $N(q) \subset \mathbb{R}^2$ the Newton polytope of q, v_0 be a vertex of $N(q)$, v_1, v_2 the adjacent vertices, indexed counterclockwise.

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So, $f(x)$ has a convergent Laurent series expansion with support contained in $x^w + \mathcal{C}$ for suitable $w \in \mathbb{Z}^2$ [GKZ], where $\mathcal{C}$ is the cone

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Algebraic hypergeometric functions in one variable

Let $v(z) := \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r}(p_i\, n)!}{\prod_{j=1}^{s}(q_j\, n)!} z^n$, $\sum_{i=1}^{r} p_i = \sum_{j=1}^{s} q_j$.

- Using Beukers-Heckman '89 it was shown by FRV '03 that v defines an algebraic function if and only the *height* $d := s - r$, equals 1 and the coefficients A_n are integral for every $n \in \mathbb{N}$.
- BH gave an explicit classification of all algebraic univariate hypergeometric series, from which [FRV, Bober] classified all integral factorial ratio sequences of height 1.
- Assume that $\gcd(p_1, \ldots, p_r, q_1, \ldots, q_{r+1}) = 1$. Then there exist three infinite families for A_n:
  1. $\frac{((a+b)\,n)!}{(a\,n)!\,(b\,n)!}$, $\gcd(a,b) = 1$,
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Theorem M

In our context, (dehomogenized) series of the form
$u(z) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r}(p_i\, n + k_i)!}{\prod_{j=1}^{s}(q_j\, n)!}\, z^n, \quad k_i \in \mathbb{N}$ occur (with $\sum_{i=1}^{r} p_i = \sum_{j=1}^{s} q_j$).

Calling $A_n = \frac{\prod_{i=1}^{r}(p_i\, n)!}{\prod_{j=1}^{s}(q_j\, n)!}$, the coefficients of u equal $h(n)A_n$, with h a polynomial.

Theorem

$$u(z) := \sum_{n\geq 0} h(n) A_n\, z^n, \qquad v(z) := \sum_{n \geq 0} A_n\, z^n.$$

(i) The series $u(z)$ is algebraic if and only if $v(z)$ is algebraic.
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So far, so good

. . . but how we figured out the statement of the general result and how to guess the corresponding statement in dimensions 3 and higher?

A-hypergeometric systems

Following [Gel′fand, Kapranov and Zelevinsky '87,'89,'90] we associate to a matrix $A \in \mathbb{Z}^{d\times n}$ and a vector $\beta \in \mathbb{C}^d$ a left ideal in the Weyl algebra in n variables:

The A*-hypergeometric system* **with parameter** β **is the left ideal** $H_A(\beta)$ **in the Weyl algebra** D_n **generated by the** *toric operators* $\partial^u - \partial^v$**, for all** $u, v \in \mathbb{N}^n$ **such that** $Au = Av$**, and the** *Euler operators* $\sum_{j=1}^n a_{ij} z_j \partial_j - \beta_i$ **for** $i = 1, \ldots, d$**.**

Note that the binomial operators generate the whole toric ideal I_A.

- The *Euler operators* impose A-homogeneity to the solutions
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Gauss functions, revisited GKZ style

Consider the configuration in $\mathbb{R}^3$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$

$$\ker_{\mathbb{Z}}(A) = \langle (1,1,-1,-1) \rangle \quad (1,1,-1,-1) = (1,1,0,0) - (0,0,1,1)$$

- The following GKZ-hypergeometric system of equations in four variables x_1, x_2, x_3, x_4 is a nice encoding for Gauss equation in one variable:

$$\begin{cases} (\partial_1\partial_2 - \partial_3\partial_4)(\varphi) &= 0 \\ (x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + x_4\partial_4)(\varphi) &= \beta_1\varphi \\ (x_2\partial_2 + x_3\partial_3)(\varphi) &= \beta_2\varphi \\ (x_2\partial_2 + x_4\partial_4)(\varphi) &= \beta_3\varphi \end{cases}$$

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Some features

- *A-hypergeometric systems are homogeneous versions of classical hypergeometric systems in $n-d$ variables ($d = \mathrm{rank}(A)$).*
- *Combinatorially defined in terms of configurations.*
- *Closely related to toric geometry.*
- *One may use algorithmic and computational techniques [Saito, Sturmfels, Takayama '99].*
- *$H_A(\beta)$ is always holonomic and it has regular singularities iff A is regular [GKZ, Adolphson, Hotta, Schulze–Walther]*
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A-hypergeometric systems

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Theorems/Conjectures about A-hypergeometric systems

GKZ-definition of multivariate hypergeometric functions gives a combinatorial meaning to parameters and a geometric meaning to solutions.

Rational A-hypergeometric functions

- We studied the constraints imposed on a regular A by the existence of stable rational A-hypergeometric functions: essentially, functions with singularities along the discriminant locus D_A.
- We proved that "general" configurations A do NOT admit such rational functions [Cattani–D.–Sturmfels '01] and gave a conjectural characterization of the configurations and of the shape of the rational functions in terms of *essential Cayley configurations* and toric residues.
- All codimension 1 configurations [CDS '01], dimension 1 [Cattani–D'Andrea–D. '99] and 2 [CDS '01], Lawrence configurations [CDS '02], fourfolds in $\mathbb{P}^r$ [Cattani–D. '04], codimension 2 [CDRV '08]

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Cayley configurations

Definition

A configuration $A \subset \mathbb{Z}^d$ is said to be a *Cayley configuration* if there exist vector configurations $A_1, \ldots, A_{k+1}$ in $\mathbb{Z}^r$ such that –up to affine equivalence–

$$A = \{e_1\} \times A_1 \cup \cdots \cup \{e_{k+1}\} \times A_{k+1} \subset \mathbb{Z}^{k+1} \times \mathbb{Z}^r, \quad (15)$$

where $e_1, \ldots, e_{k+1}$ is the standard basis of $\mathbb{Z}^{k+1}$.

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Our statement of bivariate hypergeometric series is the translation of the general combinatorial structure on the A-side (which also provides statements for the generalization to any number of variables)

The study of A-hypergeometric systems provides a general framework under which we can treat many systems that had been studied separately in the literature.

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The End

Thank you for your attention!