Hypergeometric series with algebro-geometric dressing

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Based on joint work:

The structure of bivariate rational hypergeometric functions (with Eduardo Cattani and Fernando Rodríguez Villegas) arXiv:0907.0790, to appear: IMRN.


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Aim and plan of the talk

**Aim:** Show two sample results on bivariate hypergeometric series/recurrences with inspiration/proof driven by algebraic geometry.

1. *First problem:* Solutions to hypergeometric recurrences in $\mathbb{Z}^2$.
3. *Definitions/properties concerning A-hypergeometric systems and toric residues.*
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**Aim:** Show two sample results on bivariate hypergeometric series/recurrences with inspiration/proof driven by algebraic geometry.

1. **First problem:** Solutions to hypergeometric recurrences in $\mathbb{Z}^2$.
2. **Second problem:** Characterize hypergeometric rational series in 2 variables.
3. **Definitions/properties concerning $A$-hypergeometric systems and toric residues.**
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**Aim:** Show two sample results on bivariate hypergeometric series/recurrences with inspiration/proof driven by algebraic geometry.

1. First problem: Solutions to hypergeometric recurrences in $\mathbb{Z}^2$.
3. Definitions/properties concerning A-hypergeometric systems and toric residues.
Solutions to hypergeometric recurrences

\[ A_n := \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}, \quad F(\alpha, \beta, \gamma; x) = \sum_{n \geq 0} A_n x^n. \]

\[(c)_n = c(c + 1) \ldots (c + n - 1), \quad (1)_n = n!, \quad \text{Pochhammer symbol}\]

Key equivalence

The coefficients \( A_n \) satisfy the following recurrence:

\[(1 + n)(\gamma + n)A_{n+1} - (\alpha + n)(\beta + n)A_n = 0 \quad (1)\]

(1) is equivalent to the fact that \( F(\alpha, \beta, \gamma; x) \) satisfies Gauss differential equation (Kummer, Riemann):

\[ [\Theta(\Theta + \gamma - 1) - x(\Theta + \alpha)(\Theta + \beta)](F) = 0, \quad \Theta = x \frac{d}{dx} \]
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So: \( A_{n+1}/A_n \) is the **rational** function of \( n \): \( (\alpha+n)(\beta+n)/(1+n)(\gamma+n) \).

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If we define \( A_n = 0 \) for all \( n \in \mathbb{Z}_{<0} \), the coefficients \( A_n \) satisfy the recurrence:

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\[ B_n := \frac{(\alpha)_n(\beta)_n}{(\gamma)_n(\delta)_n}, \quad \gamma, \delta \notin \mathbb{Z}_{<0}, \quad G(\alpha, \beta, \gamma, \delta; x) = \sum_{n \geq 0} B_n x^n. \]

Caveat

\[(\delta + n)(\gamma + n)B_{n+1} - (\alpha + n)(\beta + n)B_n = 0, \quad \text{for all } n \in \mathbb{N}. \quad (3)\]

but \(G(\alpha, \beta, \gamma; x)\) does not satisfy the differential equation:

\[ [(\Theta + \delta - 1)(\Theta + \gamma - 1) - x(\Theta + \alpha)(\Theta + \beta)](G) = 0. \]
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The normalization hides the initial condition

If we define \( B_n = 0 \) for all \( n \in \mathbb{Z}_{<0} \), then

\[(n+1)(\delta + n)(\gamma + n)B_{n+1} - (n+1)(\alpha + n)(\beta + n)B_n = 0, \quad \text{for all } n \in \mathbb{Z}. \tag{4}\]

\( G(\alpha, \beta, \gamma; x) \) does satisfy the differential equation:

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Let $a_{mn}$, $m, n \in \mathbb{N}$ such that there exist two rational functions $R_1(m, n)$, $R_2(m, n)$ expressible as \textit{products of (affine) linear functions} in $(m, n)$, such that
\begin{align*}
\frac{a_{m+1,n}}{a_{mn}} &= R_1(m, n), \\
\frac{a_{m,n+1}}{a_{mn}} &= R_2(m, n)
\end{align*}
(5)
(with obvious \textit{compatibility} conditions).

Write
\begin{align*}
R_1(m, n) &= \frac{P_1(m, n)}{Q_1(m+1, n)}, \\
R_2(m, n) &= \frac{P_2(m, n)}{Q_2(m, n + 1)}.
\end{align*}
Naive generalization

Let $a_{mn}$, $m, n \in \mathbb{N}$ such that there exist two rational functions $R_1(m, n)$, $R_2(m, n)$ expressible as \textit{products of (affine) linear functions} in $(m, n)$, such that

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Hypergeometric recurrences in two variables

Naive generalization, suite

Consider the generating function $F(x_1, x_2) = \sum_{m,n \in \mathbb{N}} a_{mn} x_1^m x_2^n$ and the differential operators \( (\theta_i = x_i \frac{\partial}{\partial x_i}) \):

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\Delta_1 = Q_1(\theta_1, \theta_2) - x_1 P_1(\theta_1, \theta_2) \quad \Delta_2 = Q_2(\theta_1, \theta_2) - x_2 P_2(\theta_1, \theta_2).
\]

Then, the recurrences (5) in the coefficients $a_{mn}$ are equivalent to $\Delta_1(F) = \Delta_2(F) = 0$ if $Q_1(0, n) = Q_2(m, 0) = 0$ and in this case, if we extend the definition of $a_{mn}$ by 0, the recurrences

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hold for all $(m, n) \in \mathbb{Z}^2$. 
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hold for all \((m, n) \in \mathbb{Z}^2\).
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hold for all \( (m, n) \in \mathbb{Z}^2 \).
Dissections

A subdivision of a regular \(n\)-gon into \((m + 1)\) cells by means of nonintersecting diagonals is called a dissection.

How many dissections are there?

\[
d_{m,n} = \frac{1}{m+1} \binom{n-3}{m} \binom{m+n-1}{m}; \quad 0 \leq m \leq n - 3.
\]

So, the generating function is naturally defined for \((m, n)\) belonging to the lattice points in the rational cone \(\{(a, b)/0 \leq a \leq b - 3\}\) (and 0 outside).
Two examples from combinatorics

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[Example 9.2, Gessell and Xin, *The generating function of ternary trees and continued fractions*, EJC ’06]

\[ GX(x, y) = \frac{1 - xy}{1 - xy^2 - 3xy - x^2y} = \sum_{m,n \geq 0} \binom{m+n}{2m-n} x^m y^n, \]

where \( \binom{a}{b} \) is defined as 0 if \( b < 0 \) or \( a - b < 0 \).

So we are summing over the lattice points in the convex rational cone \( \{(a, b) \in \mathbb{R}^2 : 2a - b \geq 0, 2b - a \geq 0\} = \mathbb{R}_{\geq 0}(1, 2) + \mathbb{R}_{\geq 0}(2, 1) \). Or: the terms are defined over \( \mathbb{Z}^2 \) extending by 0 outside the cone.
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Data

Consider the hypergeometric terms $a_{m,n} = (-1)^n \frac{(2m-n+2)!}{n! m! (m-2n)!}$ for $(m, n)$ integers with $m - 2n \geq 0, n \geq 0$, which satisfy the recurrences:

$$\frac{a_{m+1,n}}{a_{m,n}} = \frac{(2m - n + 4) (2m - n + 3)}{(m + 1) (m + 1 - 2n)} = \frac{P_1(m, n)}{Q_1(m + 1, n)}$$

$P_1(m, n) = (2m - n + 4) (2m - n + 3)$, $Q_1(m, n) = m (m - 2n)$

$$\frac{a_{m,n+1}}{a_{m,n}} = -\frac{(m - 2n) (m - 2n - 1)}{(2m - n + 2) (n + 1)} = \frac{P_2(m, n)}{Q_2(m, n + 1)}$$

$P_2(m, n) = -(m - 2n) (m - 2n - 1)$, $Q_2(m, n) = (2m - n + 3) n$
Our results through an example

Data

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for \((m, n)\) integers with \(m - 2n \geq 0, n \geq 0\), which satisfy the recurrences:

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\frac{a_{m+1,n}}{a_{m,n}} = \frac{(2m - n + 4) (2m - n + 3)}{(m + 1) (m + 1 - 2n)} = \frac{P_1(m, n)}{Q_1(m + 1, n)}
\]

\[ P_1(m, n) = (2m - n + 4) (2m - n + 3), \quad Q_1(m, n) = m (m - 2n) \]

\[
\frac{a_{m,n+1}}{a_{m,n}} = -\frac{(m - 2n) (m - 2n - 1)}{(2m - n + 2) (n + 1)} = \frac{P_2(m, n)}{Q_2(m, n + 1)}
\]

\[ P_2(m, n) = -(m - 2n) (m - 2n - 1), \quad Q_2(m, n) = (2m - n + 3) n \]
Our results through an example

Data

Consider the hypergeometric terms \( a_{m,n} = (-1)^n \frac{(2m-n+2)!}{n! m! (m-2n)!} \) for \((m, n)\) integers with \(m - 2n \geq 0, n \geq 0\), which satisfy the recurrences:

\[
\frac{a_{m+1,n}}{a_{m,n}} = \frac{(2m-n+4)(2m-n+3)}{(m+1)(m+1-2n)} = \frac{P_1(m,n)}{Q_1(m+1,n)}
\]

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P_1(m, n) = (2m-n+4)(2m-n+3), \quad Q_1(m, n) = m(m-2n)
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Our results through an example

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Consider the hypergeometric terms $a_{m,n} = \left(-1\right)^n \frac{(2m-n+2)!}{n! m! (m-2n)!}$ for $(m, n)$ integers with $m - 2n \geq 0, n \geq 0$, which satisfy the recurrences:

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$$
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Our results through an example

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Our results through an example

We have that the terms $t_{m,n} = a_{mn}$ for $m - 2n \geq 0, n \geq 0$ and $t_{(m,n)} = 0$ for any other $(m, n) \in \mathbb{Z}^2$, satisfy the recurrences:

$$Q_1(m+1,n)t_{m+1,n} - P_1(m,n)t_{m,n} = Q_2(m,n+1)t_{(m,n+1)} - P_2(m,n)t_{m,n} = 0.$$  
(6)

Question

Which other terms $t_{m,n}, (m, n) \in \mathbb{Z}^2$ satisfy (6)?

Remark

When the linear forms in the polynomials $P_i, Q_i$ defining the recurrences have generic constant terms, the solution is given by the Ore-Sato coefficients.
Our results through an example

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Answer

There are three other solutions \( b_{mn}, c_{mn}, d_{mn} \) (up to linear combinations)
Our results through an example

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Our results through an example

**Answer**

There are four solutions $a_{mn}, b_{mn}, c_{mn}, d_{mn}$ (up to linear combinations), with generating series $F_1, \ldots, F_4$:

$$a_{m,n} = (-1)^n \frac{(2m-n+2)!}{n! m! (m-2n)!}, \quad F_1 = \sum_{m-2n \geq 0} \sum_{n \geq 0} a_{m,n} x_1^m x_2^n,$$

$$b_{m,n} = (-1)^m \frac{(2m-n-1)!}{n! m! (-2m+n+3)!}, \quad F_2 = \sum_{-2m+n \geq 3} \sum_{m \geq 0} b_{m,n} x_1^m x_2^n,$$

$$c_{m,n} = (-1)^{m+n} \frac{(-m-1)! (-n-1)!}{(m-2n)! (-2m+n-3)!}, \quad F_3 = \sum_{m-2n \geq 0} \sum_{-2m+n \geq 3} c_{m,n} x_1^m x_2^n,$$

$$d_{-2,-1} = 1, \quad F_4 = x_1^{-2} x_2^{-1}.$$

In all cases, $t_{mn} = 0$ outside the support of the series.
Pictorially
The generating functions $F_i$ satisfy the differential equations:

\[
\Theta_1(\Theta_1 - 2\Theta_2) - x_1 (2\Theta_1 - \Theta_2 + 4)(2\Theta_1 - \Theta_2 + 3)(F) = 0,
\]

\[
\Theta_2(-2\Theta_1 + \Theta_2 - 3) - x_2 (2\Theta_2 - \Theta_1)(2\Theta_2 - \Theta_1 + 1)(F) = 0.
\]

Consider the system of binomial equations:

\[
q_1 = \partial_1^1 \partial_3^1 - \partial_2^2, \quad q_2 = \partial_2^1 \partial_4^1 - \partial_3^2
\]

in the commutative polynomial ring $\mathbb{C}[\partial_1, \ldots, \partial_4]$.

The zero set $q_1 = q_2 = 0$ has two irreducible components, one of degree 3 and multiplicity 1, which intersects $(\mathbb{C}^*)^4$ (it is the twisted cubic), and another component "at infinity": $\{\partial_2 = \partial_3 = 0\}$, of degree 1 and multiplicity $1 = \min\{2 \times 2, 1 \times 1\}$.
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\[
\begin{align*}
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This multiplicity equals the intersection multiplicity at \((0,0)\) of the system of two binomials in two variables:

\[ p_1 = \partial_3^a - \partial_2^b, \quad p_2 = \partial_2^c - \partial_3^d, \quad a = 1, b = 2, c = 1, d = 2 \]

The multiplicity of this only (non homogeneous) component at infinity is equal to the dimension of the space of solutions of the recurrences with finite support.
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Finite recurrences and polynomial solutions
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Let $B \in \mathbb{Z}^{n \times 2}$ with rows $b_1, \ldots, b_n$ satisfying $b_1 + \cdots + b_n = 0$.

\[
P_i = \prod_{b_{ji} < 0} \prod_{l=0}^{|b_{ji}| - 1} (b_j \cdot \theta + c_j - l),
\]

(7)

\[
Q_i = \prod_{b_{ji} > 0} \prod_{l=0}^{b_{ji} - 1} (b_j \cdot \theta + c_j - l), \quad \text{and}
\]

(8)

\[
H_i = Q_i - x_i P_i,
\]

(9)

where $b_j \cdot \theta = \sum_{k=1}^{2} b_{jk} \theta x_k$.

The operators $H_i$ are called Horn operators and generate the left ideal Horn $(B, c)$ in the Weyl algebra $D_2$. Call $d_i = \sum_{b_{ij} > 0} b_{ij} = - \sum_{b_{ij} < 0} b_{ij}$ the order of the operator $H_i$. 

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The operators $H_i$ are called \textit{Horn operators} and generate the left ideal \textit{Horn} $(\mathcal{B}, c)$ in the Weyl algebra $D_2$. Call $d_i = \sum_{b_{ij} > 0} b_{ij} = - \sum_{b_{ij} < 0} b_{ij}$ the \textit{order} of the operator $H_i$. 
Let $B \in \mathbb{Z}^{n \times 2}$ as above and let $A \in \mathbb{Z}^{(n-2) \times n}$ such that the columns $b^{(1)}, b^{(2)}$ of $B$ span $\ker_{\mathbb{Q}}(A)$.

Write any vector $u \in \mathbb{R}^n$ as $u = u_+ - u_-$, where $(u_+)_i = \max(u_i, 0)$, and $(u_-)_i = -\min(u_i, 0)$.

**Definition**

$$T_i = \partial b^{(i)}_+ - \partial b^{(i)}_-,$$ 

$i = 1, 2$.

The left $D_n$-ideal $H_B(c)$ is defined by:

$$H_B(c) = \langle T_1, T_2 \rangle + \langle A \cdot \theta - A \cdot c \rangle \subseteq D_n.$$
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Theorem

[D.- Matusevich - Sadykov ’05] For generic complex parameters $c_1, \ldots, c_n$, the ideals $\text{Horn}(B, c)$ and $H_B(c)$ are holonomic. Moreover,

$$\text{rank}(H_B(c)) = \text{rank}(\text{Horn}(B, c)) = d_1 d_2 - \sum_{b_i, b_j \text{ depdt}} \nu_{ij} = g \cdot \text{vol}(A) + \sum_{b_i, b_j \text{ indepdt}} \nu_{ij} ,$$

where the pairs $b_i, b_j$ of rows lie in opposite open quadrants of $\mathbb{Z}^2$.

Remarks

Solutions to recurrences with finite support correspond to (Laurent) polynomial solutions. These solutions come from (non homogeneous) primary components at infinity of the binomial ideal $\langle T_1, T_2 \rangle$. There are many linearly independent. For special parameters a special study is needed, along the lines in [D. - Matusevich and Miller ’10].
**Theorem**

[D.- Matusevich - Sadykov ’05] For generic complex parameters \( c_1, \ldots, c_n \), the ideals \( \text{Horn}(\mathcal{B}, c) \) and \( H_{\mathcal{B}}(c) \) are holonomic. Moreover,

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**Remarks**

Solutions to recurrences with finite support correspond to (Laurent) polynomial solutions. These solutions come from (non homogeneous) primary components at infinity of the binomial ideal \(\langle T_1, T_2 \rangle\). There are \(\sum \nu_{ij}\) many linearly independent. For special parameters a special study is needed, along the lines in [D. - Matusevich and Miller ’10].
Theorem

[D.- Matusevich - Sadykov ’05] For generic complex parameters $c_1, \ldots, c_n$, the ideals $\text{Horn}(\mathcal{B}, c)$ and $H_{\mathcal{B}}(c)$ are holonomic. Moreover,

$$\text{rank}(H_{\mathcal{B}}(c)) = \text{rank}((\text{Horn})(\mathcal{B}, c)) = d_1d_2 - \sum_{\begin{subarray}{c} b_i, b_j \text{depdt} \end{subarray}} \nu_{ij} = g \cdot \text{vol}(A) + \sum_{\begin{subarray}{c} b_i, b_j \text{indepdt} \end{subarray}} \nu_{ij},$$

where the pairs $b_i, b_j$ of rows lie in opposite open quadrants of $\mathbb{Z}^2$.

Remarks

Solutions to recurrences with finite support correspond to (Laurent) polynomial solutions. These solutions come from (non homogeneous) primary components at infinity of the binomial ideal $\langle T_1, T_2 \rangle$. There are $\sum \nu_{ij}$ many linearly independent. For special parameters a special study is needed, along the lines in [D. - Matusevich and Miller ’10].
General philosophy

Moral of this story

Key to the answer is the homogenization and translation to the A-side!
Moral of this story

Key to the answer it the homogenization and translation to the $A$-side!
Examples of rational bivariate hypergeometric series

The proof in the talk!

Lemma: The series $f(s_1,s_2)(x) := \sum_{m\in\mathbb{N}^2} \frac{(s_1m_1+s_2m_2)!}{(s_1m_1)!(s_2m_2)!} x_1^{m_1} x_2^{m_2}$ is a rational function for all $(s_1, s_2) \in \mathbb{N}^2$.

Proof: $f(0,0)(x_1, x_2) = \sum_{m\in\mathbb{N}^2} x_1^{m_1} x_2^{m_2} = \frac{1}{(1-x_1)(1-x_2)}$,

$f(1,1)(x) = \sum_{m\in\mathbb{N}^2} \frac{(m_1+m_2)!}{m_1! m_2!} x_1^{m_1} x_2^{m_2} = \frac{1}{1-x_1-x_2}$,

$f(2,2)(x_1^2, x_2^2) = \sum_{m\in\mathbb{N}^2} \frac{(2m_1+2m_2)!}{(2m_1)!(2m_2)!} x_1^{2m_1} x_2^{2m_2} =$

$$\frac{1}{4} (f(1,1)(x_1, x_2) + f(1,1)(-x_1, x_2) + f(1,1)(x_1, -x_2) + f(1,1)(-x_1, -x_2)) =$$

$$\frac{1-x_1^2-x_2^2}{1-2x_1^2-2x_2^2-2x_1^2x_2+x_1^4+x_2^4},$$

$$f(2,2)(x_1, x_2) = \frac{1-x_1-x_2}{1-2x_1-2x_2-2x_1x_2+x_1^2+x_2^2} \cdot \Box$$
Examples of rational bivariate hypergeometric series

The proof in the talk!

**Lemma:** The series \( f_{(s_1,s_2)}(x) := \sum_{m \in \mathbb{N}^2} \frac{(s_1 m_1 + s_2 m_2)!}{(s_1 m_1)!(s_2 m_2)!} x_1^{m_1} x_2^{m_2} \) is a rational function for all \((s_1, s_2) \in \mathbb{N}^2\).

**Proof:**

\[
\begin{align*}
  f_{(0,0)}(x_1, x_2) &= \sum_{m \in \mathbb{N}^2} x_1^{m_1} x_2^{m_2} = \frac{1}{(1-x_1)(1-x_2)}, \\
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  \frac{1}{4} \left( f_{(1,1)}(x_1, x_2) + f_{(1,1)}(-x_1, x_2) + f_{(1,1)}(x_1, -x_2) + f_{(1,1)}(-x_1, -x_2) \right) = \\
  \frac{1}{1-x_1^2-x_2^2} \cdot \\
  \frac{1}{1-2x_1^2-2x_2^2-2x_1^2 x_2+x_1^4+x_2^4}, \\
  f_{(2,2)}(x_1, x_2) &= \frac{1-x_1-x_2}{1-2x_1-2x_2-2x_1 x_2+x_1^2+x_2^2}. \blacksquare
\end{align*}
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Examples of rational bivariate hypergeometric series

The proof in the talk!

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f(0,0)(x_1, x_2) &= \sum_{m \in \mathbb{N}^2} x_1^{m_1} x_2^{m_2} = \frac{1}{(1-x_1)(1-x_2)}, \\
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&= \frac{1}{4} (f(1,1)(x_1, x_2) + f(1,1)(-x_1, x_2) + f(1,1)(x_1, -x_2) + f(1,1)(-x_1, -x_2)) = \frac{1-x_1^2-x_2^2}{1-2x_1^2-2x_2^2-2x_1^2 x_2+x_1^2+x_2^2}, \\
f(2,2)(x_1, x_2) &= \frac{1-x_1-x_2}{1-2x_1-2x_2-2x_1 x_2+x_1^2+x_2^2}. \Box
\end{align*}$
Examples of rational bivariate hypergeometric series

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Examples of rational bivariate hypergeometric series

The proof in the talk!

**Lemma:** The series $f_{(s_1,s_2)}(x) := \sum_{m \in \mathbb{N}^2} \frac{(s_1 m_1 + s_2 m_2)!}{(s_1 m_1)! (s_2 m_2)!} x_1^{m_1} x_2^{m_2}$.

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- $f_{(0,0)}(x_1, x_2) = \sum_{m \in \mathbb{N}^2} x_1^{m_1} x_2^{m_2} = \frac{1}{(1-x_1)(1-x_2)}$,

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Examples of rational bivariate hypergeometric series

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**Lemma:** The series \( f(s_1, s_2)(x) := \sum_{m \in \mathbb{N}^2} \frac{(s_1 m_1 + s_2 m_2)!}{(s_1 m_1)! (s_2 m_2)!} x_1^{m_1} x_2^{m_2} \).

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**Lemma:** The series
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**Proof:**

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A second proof!

**Proof:** The series $f_{(s_1,s_2)}(x) := \sum_{m\in \mathbb{N}^2} \frac{(s_1m_1+s_2m_2)!}{(s_1m_1)!(s_2m_2)!} x_1^{m_1} x_2^{m_2}$ defines a rational function for all $(s_1, s_2) \in \mathbb{N}^2$ because it equals the following residue:

$$f_{(s_1,s_2)}(x) = \sum_{\xi_1^{s_1} = -x_1, \xi_2^{s_2} = -x_2} \text{Res}_{\xi} \left( \frac{t_1^{s_1} t_2^{s_2}/(t_1 + t_2 + 1)}{(x_1 + t_1^{s_1})(x_2 + t_2^{s_2})} \frac{dt_1}{t_1} \wedge \frac{dt_2}{t_2} \right) =$$

$$= \frac{1}{s_1 s_2} \sum_{\xi_1^{s_1} = -x_1, \xi_2^{s_2} = -x_2} \frac{1}{\xi_1 + \xi_2 + 1} \diamond$$
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\]

\[
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Question

When is a hypergeometric series in 2 variables rational?

Let \( c^i = (c_1^i, c_2^i) \) and \( d^j = (d_1^j, d_2^j) \) for \( i = 1, \ldots, r; j = 1, \ldots, s \) be vectors in \( \mathbb{N}^2 \). When is the series

\[
\sum_{m \in \mathbb{N}^2} \frac{\prod_{i=1}^r (c_1^i m_1 + c_2^i m_2)!}{\prod_{j=1}^s (d_1^j m_1 + d_2^j m_2)!} x_1^{m_1} x_2^{m_2}
\]

the Taylor expansion of a rational function?
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Rational bivariate hypergeometric series

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Theorem:

Let $c^i = (c^i_1, c^i_2)$ and $d^j = (d^j_1, d^j_2)$ for $i = 1, \ldots, r; j = 1, \ldots, s$ be vectors in $\mathbb{N}^2$ (with $\sum c^i = \sum d^j$).

The series $\sum_{m \in \mathbb{N}^2} \frac{\prod_{i=1}^r (c^i_1 m_1 + c^i_2 m_2)!}{\prod_{j=1}^s (d^j_1 m_1 + d^j_2 m_2)!} x_1^{m_1} x_2^{m_2}$ is the Taylor expansion of a rational function if and only if it is of the form $f_{(s_1, s_2)}(x)$. 

A. Dickenstein (U. Buenos Aires)
Rational bivariate hypergeometric series

Answer

Theorem:

Let \( c^i = (c_1^i, c_2^i) \) and \( d^j = (d_1^j, d_2^j) \) for \( i = 1, \ldots, r; j = 1, \ldots, s \) be vectors in \( \mathbb{N}^2 \) (with \( \sum c^i = \sum d^j \)).

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**Theorem:**

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Let $c^i = (c_1^i, c_2^i)$ and $d^j = (d_1^j, d_2^j)$ for $i = 1, \ldots, r; j = 1, \ldots, s$ be vectors in $\mathbb{N}^2$ (with $\sum c^i = \sum d^j$).

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Gessell and Xin´s example of a rational bivariate hypergeometric series

What if the cone is not the first orthant?

We had

\[ G(x, y) = \frac{1 - xy}{1 - xy^2} = \sum \binom{m + n}{3m - n} x^m y^n \]

where we are summing over the lattice points in the (pointed) non-unimodular convex cone \( \mathbb{R}_{\geq 0} (1, 2) + \mathbb{R}_{\geq 0} (2, 1) \).

Calling \( m_1 = m - n, m_2 = n - m \) (so that \( m = \frac{m_1 + m_2}{3}, n = \frac{m_1 + 2m_2}{3} \)),

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where \( L = \mathbb{Z}(1, 2) + \mathbb{Z}(2, 1) = \{ (m_1, m_2) : m_1 \equiv m_2 \mod 3 \} \) and \( x = x^1, y = y^2 \).

The shape of the non-zero coefficients is the expected, but the sum is over a sublattice.
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\[ \frac{1 - xy}{1 - xy^2 - 3xy - x^2y} = \sum (m_1, m_2) \in L \cap \mathbb{N}^2 \frac{(m_1 + m_2)!}{m_1! m_2!} u_1^{m_1} u_2^{m_2} , \]

where \( L = \mathbb{Z}(1, 2) + \mathbb{Z}(2, 1) = \{(m_1, m_2) \in \mathbb{Z}^2 : m_1 \equiv m_2 \mod 3 \} \) and \( u_1^3 = x^2y, u_2^3 = xy^2 \).

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\frac{1 - xy}{1 - xy^2 - 3xy - x^2y} = \sum_{(m_1, m_2) \in L \cap \mathbb{N}^2} \frac{(m_1 + m_2)!}{m_1! m_2!} u_1^{m_1} u_2^{m_2},
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The shape of the non zero coefficients is the expected, but the sum is over a sublattice.
Gessell and Xin’s example of a rational bivariate hypergeometric series

What if the cone is not the first orthant?

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\[ GX(x, y) = \frac{1 - xy}{1 - xy^2 - 3xy - x^2y} = \sum \binom{m + n}{2m - n} x^m y^n , \]

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The shape of the non zero coefficients is the expected, but the sum is over a sublattice.
The general result

Data

Suppose we are given linear functionals
\[ \ell_i(m_1, m_2) := \langle b_i, (m_1, m_2) \rangle + k_i, \quad i = 1, \ldots, n, \]
where \( b_i \in \mathbb{Z}^2 \setminus \{0\}, k_i \in \mathbb{Z} \) and \( \sum_{i=1}^{n} b_i = 0 \).

Take \( C \) a rational convex cone. The bivariate series:
\[ \phi(x_1, x_2) = \sum_{m \in C \cap \mathbb{Z}^2} \frac{\prod_{\ell_i(m) < 0} (-1)^{\ell_i(m)} (-\ell_i(m) - 1)!}{\prod_{\ell_j(m) > 0} \ell_j(m)!} x_1^{m_1} x_2^{m_2}. \tag{10} \]
is called a Horn series.

The coefficients \( c_m \) of \( \phi \) satisfy hypergeometric recurrences: for \( j = 1, 2 \), and any \( m \in C \cap \mathbb{Z}^2 \) such that \( m + e_j \) also lies in \( C \):
\[ \frac{c_{m+e_j}}{c_m} = \frac{\prod_{b_{ij} < 0} \prod_{l=0}^{b_{ij}+1} \ell_i(m) - l}{\prod_{b_{ij} > 0} \prod_{l=1}^{b_{ij}} \ell_i(m) + l}. \]
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**Theorem**

[Cattani, D.-, R. Villegas '09]

*If the Horn series $\phi(x_1, x_2)$ is a rational function then: either*

(i) $n = 2r$ is even and, after reordering we may assume:

$$b_1 + b_{r+1} = \cdots = b_r + b_{2r} = 0, \text{ or}$$

(ii) $B$ consists of $n = 2r + 3$ vectors and, after reordering, we may assume that $b_1, \ldots, b_{2r}$ satisfy (11) and $b_{2r+1} = s_1 \nu_1$, $b_{2r+2} = s_2 \nu_2$, $b_{2r+3} = -b_{2r+1} - b_{2r+2}$, where $\nu_1, \nu_2$ are the primitive, integral, inward-pointing normals of $C$ and $s_1, s_2$ are positive integers.

Moreover, $\phi$ can be expressed as a *residue.*
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Moreover, $\phi$ can be expressed as a residue.
Gessell and Xin´s example as a residue

\[ \phi(x) = GX(-x) = \sum_{m \in \mathbb{C} \cap \mathbb{Z}^2} (-1)^{m_1 + m_2} \left( m_1 + m_2 \right) \left( 2m_1 - m_2 \right) x_1^{m_1} x_2^{m_2} \] is a Horn series.

We read the lattice vectors \( b_1 = (-1, -1), b_2 = (-1, 2), b_3 = (2, -1), \) and we enlarge them to a configuration \( B \) by adding the vectors \( b_4 = (1, 0) \) and \( b_5 = (-1, 0). \)

\( B \) is the Gale dual of the configuration \( A \):

\[
A = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
0 & 1 & 2 & 0 & 3
\end{pmatrix}
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and \( \phi(x) \) is the dehomogenization of a toric residue associated to \( f_1 = z_1 + z_2 t + z_3 t^2, f_2 = z_4 + z_5 t^3. \)

In inhomogeneous coordinates we have the not so nice expression:

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\phi(x) = \sum_{\eta^3 = -x_2/x_1} \text{Res}_\eta \left( \frac{x_2 t/(x_2 + x_2 t - t^2)}{x_2 + x_1 t^3} \right) dt,
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Outline of the proof

A key lemma about Laurent expansions of rational functions + a nice ingredient: the *diagonals* of a rational bivariate power series define classical hypergeometric algebraic univariate functions. [Polya ’22, Furstenberg ’67, Safonov ’00].

Number theoretic + monodromy ingredients: we use Theorem M below to reduce to the algebraic hyperg. functions classified by [Beukers-Heckmann ’89], [F. R. Villegas ’03, Bober ’08]

Many previous results on $A$-hypergeometric functions, allow us to analyze the possible Laurent expansions of rational hypergeometric solutions and to construct rational solutions using toric residues. [Saito-Sturmfels-Takayama ´99; Cattani, D.-Sturmfels ’01, 02; Cattani - D. ´04].
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Given a bivariate power series

\[ f(x_1, x_2) := \sum_{n,m \geq 0} a_{m,n} x_1^m x_2^n \]  

and \( \delta = (\delta_1, \delta_2) \in \mathbb{Z}^2_{>0} \), with \( \gcd(\delta_1, \delta_2) = 1 \), we define the \( \delta \)-diagonal of \( f \) as:

\[ f_\delta(t) := \sum_{r \geq 0} a_{\delta_1 r, \delta_2 r} t^r. \]
Diagonals of bivariate series

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If the series defines a rational function, then for every \( \delta = (\delta_1, \delta_2) \in \mathbb{Z}_{>0}^2 \), with \( \gcd(\delta_1, \delta_2) = 1 \), the \( \delta \)-diagonal \( f_\delta(t) \) is algebraic.
Laurent series of rational functions

Let \( p, q \in \mathbb{C}[x_1, x_2] \) coprime, \( f = p/q \), \( N(q) \subset \mathbb{R}^2 \) the Newton polytope of \( q \), \( v_0 \) be a vertex of \( N(q) \), \( v_1, v_2 \) the adjacent vertices, indexed counterclockwise.

Hence, \( N(q) \subset v_0 + \mathbb{R}_{>0} \cdot (v_1 - v_0) + \mathbb{R}_{>0} \cdot (v_2 - v_0) \).

So, \( f(x) \) has a convergent Laurent series expansion with support contained in \( x^w + C \) for suitable \( w \in \mathbb{Z}^2 \) [GKZ], where \( C \) is the cone

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C = \mathbb{R}_{\geq 0} (v_1 - v_0) + \mathbb{R}_{\geq 0} (v_2 - v_0).
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Key Lemma

The support of the series is not contained in any subcone of the form \( x^{w'} + C' \), with \( C' \) is properly contained in \( C \).
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Laurent series of rational functions

Let \( p, q \in \mathbb{C}[x_1, x_2] \) coprime, \( f = p/q \), \( N(q) \subset \mathbb{R}^2 \) the Newton polytope of \( q \), \( v_0 \) be a vertex of \( N(q) \), \( v_1, v_2 \) the adjacent vertices, indexed counterclockwise.

Hence, \( N(q) \subset v_0 + \mathbb{R}_{>0} \cdot (v_1 - v_0) + \mathbb{R}_{>0} \cdot (v_2 - v_0) \).

So, \( f(x) \) has a convergent Laurent series expansion with support contained in \( x^w + \mathcal{C} \) for suitable \( w \in \mathbb{Z}^2 \) [GKZ], where \( \mathcal{C} \) is the cone

\[
\mathcal{C} = \mathbb{R}_{\geq 0} (v_1 - v_0) + \mathbb{R}_{>0} (v_2 - v_0).
\]

Key Lemma

The support of the series is not contained in any subcone of the form \( x^{w'} + \mathcal{C}' \), with \( \mathcal{C}' \) is properly contained in \( \mathcal{C} \).
Laurent series of rational functions

Let \( p, q \in \mathbb{C}[x_1, x_2] \) coprime, \( f = p/q \), \( N(q) \subset \mathbb{R}^2 \) the Newton polytope of \( q \), \( v_0 \) be a vertex of \( N(q) \), \( v_1, v_2 \) the adjacent vertices, indexed counterclockwise.

Hence, \( N(q) \subset v_0 + \mathbb{R}_{>0} \cdot (v_1 - v_0) + \mathbb{R}_{>0} \cdot (v_2 - v_0) \).

So, \( f(x) \) has a convergent Laurent series expansion with support contained in \( x^w + C \) for suitable \( w \in \mathbb{Z}^2 \) [GKZ], where \( C \) is the cone

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C = \mathbb{R}_{\geq 0} \cdot (v_1 - v_0) + \mathbb{R}_{\geq 0} \cdot (v_2 - v_0).
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Key Lemma

The support of the series is not contained in any subcone of the form \( x^{w'} + C' \), with \( C' \) is properly contained in \( C \).
Laurent series of rational functions

Let $p, q \in \mathbb{C}[x_1, x_2]$ coprime, $f = p/q$, $N(q) \subset \mathbb{R}^2$ the Newton polytope of $q$, $v_0$ be a vertex of $N(q)$, $v_1, v_2$ the adjacent vertices, indexed counterclockwise. Hence, $N(q) \subset v_0 + \mathbb{R}_{>0} \cdot (v_1 - v_0) + \mathbb{R}_{>0} \cdot (v_2 - v_0)$.

So, $f(x)$ has a convergent Laurent series expansion with support contained in $x^w + C$ for suitable $w \in \mathbb{Z}^2$ [GKZ], where $C$ is the cone

$$C = \mathbb{R}_{\geq 0} (v_1 - v_0) + \mathbb{R}_{\geq 0} (v_2 - v_0).$$

Key Lemma

The support of the series is not contained in any subcone of the form $x^{w'} + C'$, with $C'$ is properly contained in $C$. 
Let $v(z) := \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r'} (p_i n)!}{\prod_{j=1}^{s} (q_j n)!} \ z^n$, $\sum_{i=1}^{r} p_i = \sum_{j=1}^{s} q_j$.

Using Beukers-Heckman ’89 it was shown by FRV ’03 that $v$ defines an algebraic function if and only the height $d := s - r$, equals 1 and the coefficients $A_n$ are integral for every $n \in \mathbb{N}$.

BH gave an explicit classification of all algebraic univariate hypergeometric series, from which [FRV, Bober] classified all integral factorial ratio sequences of height 1.

Assume that $\gcd(p_1, \ldots, p_r, q_1, \ldots, q_{r+1}) = 1$. Then there exist three infinite families for $A_n$:

1. $\frac{(a+b)n)!}{(a)n!(b)n!}, \quad \gcd(a, b) = 1$,
2. $\frac{(2(a+b)n)!}{((a+b)n)!((2b)n)!} \frac{(b)n!}{(a)n!(2b)n!}, \quad \gcd(a, b) = 1$,
3. $\frac{(a)n!}{(a)n!((a+b)n)!} \frac{(2b)n!}{(2b)n!((a+b)n)!}, \quad \gcd(a, b) = 1$,

and 52 sporadic cases.
Let \( v(z) := \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r'} (p_i n)!}{\prod_{j=1}^{s} (q_j n)!} \ z^n, \ \sum_{i=1}^{r} p_i = \sum_{j=1}^{s} q_j \).

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  1. \( \frac{((a+b) n)!}{(a n)! (b n)!} ; \quad \gcd(a, b) = 1 \),
  2. \( \frac{(2(a+b) n)!}{((a+b) n)! (2 b n)!} ; \quad \gcd(a, b) = 1 \),
  3. \( \frac{(2 a n)!}{(a n)! (2 b n)!} ; \quad \gcd(a, b) = 1 \),

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Algebraic hypergeometric functions in one variable

Let \( v(z) := \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} (p_{i} n)!}{\prod_{j=1}^{s} (q_{j} n)!} z^{n}, \) \( \sum_{i=1}^{r} p_{i} = \sum_{j=1}^{s} q_{j}. \)

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  2. \( \frac{(2(a+b) n)! \cdot (b n)!}{((a+b) n)! (2 b n)! (a n)!}, \) \( \gcd(a, b) = 1, \)
  3. \( \frac{(2 an)! \cdot (2 b n)!}{(a n)! (b n)! ((a+b) n)!}, \) \( \gcd(a, b) = 1, \)
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Let \( v(z) := \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} (p_i n)!}{\prod_{j=1}^{s} (q_j n)!} z^n, \sum_{i=1}^{r} p_i = \sum_{j=1}^{s} q_j \).

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  3. \( \frac{(2an)! (2b n)!}{(an)! (b n)! ((a+b) n)!}, \gcd(a, b) = 1 \),
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and 52 sporadic cases.
Theorem M

In our context, (dehomogenized) series of the form

\[ u(z) = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{r} (p_i n + k_i)!}{\prod_{j=1}^{s} (q_j n)!} z^n, \quad k_i \in \mathbb{N} \text{ occur (with } \sum_{i=1}^{r} p_i = \sum_{j=1}^{s} q_j). \]

Calling \( A_n = \frac{\prod_{i=1}^{r} (p_i n)!}{\prod_{j=1}^{s} (q_j n)!} \), the coefficients of \( u \) equal \( h(n)A_n \), with \( h \) a polynomial.

(u) The series \( u(z) \) is algebraic if and only if \( v(z) \) is algebraic.

(ii) If \( u \) is rational then \( A_n = 1 \) for all \( n \) and \( v(z) = \frac{1}{1-z} \).

Proof uses monodromy as well as number theoretic arguments.
Theorem M

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Theorem

\[ u(z) := \sum_{n \geq 0} h(n)A_n z^n, \quad v(z) := \sum_{n \geq 0} A_n z^n, \]

(i) The series \( u(z) \) is algebraic if and only if \( v(z) \) is algebraic.
(ii) If \( u \) is rational then \( A_n = 1 \) for all \( n \) and \( v(z) = \frac{1}{1-z} \).

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So far, so good

...but how we figured out the statement of the general result and how to guess the corresponding statement in dimensions 3 and higher?
Following [Gel’fand, Kapranov and Zelevinsky ’87,’89,’90] we associate to a matrix \( A \in \mathbb{Z}^{d \times n} \) and a vector \( \beta \in \mathbb{C}^d \) a left ideal in the Weyl algebra in \( n \) variables:

The \( A \)-hypergeometric system with parameter \( \beta \) is the left ideal \( H_A(\beta) \) in the Weyl algebra \( D_n \) generated by the toric operators \( \partial^u - \partial^v \), for all \( u, v \in \mathbb{N}^n \) such that \( Au = Av \), and the Euler operators \( \sum_{j=1}^{n} a_{ij} z_j \partial_j - \beta_i \) for \( i = 1, \ldots, d \).

Note that the binomial operators generate the whole toric ideal \( I_A \).

- The Euler operators impose \( A \)-homogeneity to the solutions
- The toric operators impose recurrences on the coefficients of (Puiseux) series solutions.
**A-hypergeometric systems**

Following [Gel’fand, Kapranov and Zelevinsky ’87,’89,’90] we associate to a matrix $A \in \mathbb{Z}^{d \times n}$ and a vector $\beta \in \mathbb{C}^d$ a left ideal in the Weyl algebra in $n$ variables:

The **$A$-hypergeometric system with parameter $\beta$** is the left ideal $H_A(\beta)$ in the Weyl algebra $D_n$ generated by the **toric operators** $\partial^u - \partial^v$, for all $u, v \in \mathbb{N}^n$ such that $Au = Av$, and the **Euler operators** $\sum_{j=1}^n a_{ij}z_j \partial_j - \beta_i$ for $i = 1, \ldots, d$.

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Note that the binomial operators generate the whole toric ideal $I_A$.

- The Euler operators impose $A$-homogeneity to the solutions
- The toric operators impose recurrences on the coefficients of (Puiseux) series solutions.
Gauss functions, revisited \textbf{GKZ} style

Consider the configuration in $\mathbb{R}^3$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}.$$ 

$$\ker_{\mathbb{Z}}(A) = \langle (1, 1, -1, -1) \rangle$$

$$(1, 1, -1, -1) = (1, 1, 0, 0) - (0, 0, 1, 1)$$

The following GKZ-hypergeometric system of equations in four variables $x_1, x_2, x_3, x_4$ is a nice encoding for Gauss equation in one variable:

$$\begin{cases} 
    (\partial_1 \partial_2 - \partial_3 \partial_4)(\varphi) = 0 \\
    (x_1 \partial_1 + x_2 \partial_2 + x_3 \partial_3 + x_4 \partial_4)(\varphi) = \beta_1 \varphi \\
    (x_2 \partial_2 + x_3 \partial_3)(\varphi) = \beta_2 \varphi \\
    (x_2 \partial_2 + x_4 \partial_4)(\varphi) = \beta_3 \varphi
\end{cases}$$
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Consider the configuration in \( \mathbb{R}^3 \)

\[
A = \begin{pmatrix}
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\end{pmatrix}.
\]

\( \ker_{\mathbb{Z}}(A) = \langle (1, 1, -1, -1) \rangle \) \( (1, 1, -1, -1) = (1, 1, 0, 0) - (0, 0, 1, 1) \)

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(\partial_1 \partial_2 - \partial_3 \partial_4) (\varphi) = 0 \\
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\end{cases}
\]
Gauss functions, revisited **GKZ** style

\[
\begin{align*}
\left( \partial_1 \partial_2 - \partial_3 \partial_4 \right) (\varphi) &= 0 \\
\left( x_1 \partial_1 + x_2 \partial_2 + x_3 \partial_3 + x_4 \partial_4 \right) (\varphi) &= \beta_1 \varphi \\
\left( x_2 \partial_2 + x_3 \partial_3 \right) (\varphi) &= \beta_2 \varphi \\
\left( x_2 \partial_2 + x_4 \partial_4 \right) (\varphi) &= \beta_3 \varphi 
\end{align*}
\]

(14)

Given any \((\beta_1, \beta_2, \beta_3)\) and \(v \in \mathbb{C}^n\) such that \(A \cdot v = (\beta_1, \beta_2, \beta_3)\) and \(v_1 = 0\), any solution \(\varphi\) of (14) can be written as

\[\varphi(x) = x^v f \left( \frac{x_1 x_2}{x_3 x_4} \right),\]

where \(f(z)\) satisfies Gauss equation with

\[\alpha = v_2, \ \beta = v_3, \ \gamma = v_4 + 1.\]
Given any \((\beta_1, \beta_2, \beta_3)\) and \(\nu \in \mathbb{C}^n\) such that \(A \cdot \nu = (\beta_1, \beta_2, \beta_3)\) and \(\nu_1 = 0\), any solution \(\varphi\) of (14) can be written as

\[
\varphi(x) = x^{\nu} f \left( \frac{x_1 x_2}{x_3 x_4} \right),
\]

where \(f(z)\) satisfies Gauss equation with

\[
\alpha = \nu_2, \quad \beta = \nu_3, \quad \gamma = \nu_4 + 1.
\]
Gauss functions, revisited **GKZ** style

\[
\begin{align*}
(\partial_1 \partial_2 - \partial_3 \partial_4) (\varphi) &= 0 \\
(x_1 \partial_1 + x_2 \partial_2 + x_3 \partial_3 + x_4 \partial_4) (\varphi) &= \beta_1 \varphi \\
(x_2 \partial_2 + x_3 \partial_3) (\varphi) &= \beta_2 \varphi \\
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\end{align*}
\]

Given any \((\beta_1, \beta_2, \beta_3)\) and \(\mathbf{v} \in \mathbb{C}^n\) such that \(A \cdot \mathbf{v} = (\beta_1, \beta_2, \beta_3)\) and \(v_1 = 0\), any solution \(\varphi\) of (14) can be written as

\[
\varphi(x) = x^\mathbf{v} f \left( \frac{x_1 x_2}{x_3 x_4} \right),
\]

where \(f(z)\) satisfies Gauss equation with

\[
\alpha = v_2, \quad \beta = v_3, \quad \gamma = v_4 + 1.
\]
A-hypergeometric systems

Some features

- A-hypergeometric systems are homogeneous versions of classical hypergeometric systems in \( n - d \) variables (\( d = \text{rank}(A) \)).
- Combinatorially defined in terms of configurations.
- Closely related to toric geometry.
- One may use algorithmic and computational techniques [Saito, Sturmfels, Takayama ’99].
- \( H_A(\beta) \) is always holonomic and it has regular singularities iff \( A \) is regular [GKZ, Adolphson, Hotta, Schulze–Walther].
- The singular locus of the hypergeometric \( D_n \)-module \( D_n/H_A(\beta) \) equals the zero locus of the principal \( A \)-determinant \( E_A \), whose irreducible factors are the sparse discriminants \( D_A' \) corresponding to the facial subsets \( A' \) of \( A \) [GKZ] including \( D_A \).
$A$-hypergeometric systems

Some features

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- $H_A(\beta)$ is always **holonomic** and it has regular singularities iff $A$ is regular [GKZ, Adolphson, Hotta, Schulze–Walther].
- The **singular locus** of the hypergeometric $D_n$-module $D_n / H_A(\beta)$ equals the zero locus of the **principal $A$-determinant** $E_A$, whose irreducible factors are the **sparse discriminants** $D_{A'}$ corresponding to the **facial subsets** $A'$ of $A$ [GKZ] **including** $D_A$. 
A-hypergeometric systems

Some features

- A-hypergeometric systems are *homogeneous versions* of classical hypergeometric systems *in* \(n - d\) variables (\(d = \text{rank}(A)\)).

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- The *singular locus* of the hypergeometric \(D_n\)-module \(D_n/H_A(\beta)\) equals the zero locus of the *principal* \(A\)-determinant \(E_A\), whose irreducible factors are the *sparse discriminants* \(D_{A'}\) corresponding to the *facial subsets* \(A'\) of \(A\) [GKZ] including \(D_A\).
**A-hypergeometric systems**

**Some features**

- **A-hypergeometric systems** are **homogeneous versions** of **classical hypergeometric systems in** \( n - d \) **variables** \((d = \text{rank}(A))\).

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- **Closely related to toric geometry.**

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- \( H_A(\beta) \) is always **holonomic** and it has regular singularities iff \( A \) is regular \([\text{GKZ, Adolphson, Hotta, Schulze–Walther}]\).

- The singular locus of the hypergeometric \( D_n \)-module \( D_n / H_A(\beta) \) equals the **zero locus** of the **principal A-determinant** \( E_A \), whose irreducible factors are the **sparse discriminants** \( D_{A'} \) corresponding to the **facial subsets** \( A' \) of \( A \) \([\text{GKZ}]\) **including** \( D_A \).
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A. Dickenstein (U. Buenos Aires)
Theorems/Conjectures about $A$-hypergeometric systems

GKZ-definition of multivariate hypergeometric functions gives a combinatorial meaning to parameters and a geometric meaning to solutions.

Rational $A$-hypergeometric functions

- We studied the constraints imposed on a regular $A$ by the existence of stable rational $A$-hypergeometric functions, essentially functions with singularities along the discriminant locus $D$.
- We proved that “general” configurations do NOT admit such rational functions [Cattani–D.–Sturmfels '01] and gave a conjectural characterization of the configurations and of the shape of the rational functions in terms of essential Cayley configurations and toric residues.
- All codimension 1 configurations [CDS '01], dimension 1 Lawrence configurations [CDS '02] and 2 [CDS '01], Lawrence configurations [CDRV '09] for families in $\mathbb{P}^7$ [Cattani–D. '04], codimension 2, always...
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A configuration $A \subset \mathbb{Z}^d$ is said to be a Cayley configuration if there exist vector configurations $A_1, \ldots, A_{k+1}$ in $\mathbb{Z}^r$ such that –up to affine equivalence–

$$A = \{e_1\} \times A_1 \cup \cdots \cup \{e_{k+1}\} \times A_{k+1} \subset \mathbb{Z}^{k+1} \times \mathbb{Z}^r,$$

(15)

where $e_1, \ldots, e_{k+1}$ is the standard basis of $\mathbb{Z}^{k+1}$.

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For a codimension-two essential Cayley configuration \( A \), \( r \) of the configurations \( A_i \), say \( A_1, \ldots, A_r \), must consist of two vectors and the remaining one, \( A_{r+1} \), must consist of three vectors.

To an essential Cayley configuration we associate a family of \( r + 1 \) generic polynomials in \( r \) variables with supports \( A_i \), such that any \( r \) of them intersect in a positive number of points. Adding local residues over this points gives a rational function!
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Summarizing

Our statement of bivariate hypergeometric series is the translation of the general combinatorial structure on the $A$-side (which also provides statements for the generalization to any number of variables).

The study of $A$-hypergeometric systems provides a general framework under which we can treat many systems that had been studied separately in the literature.

Questions

- Describe all algebraic Laurent series solutions for Cayley configurations (in progress).
- How to prove the conjectures beyond dimension/codimension two? There exists a characterization of normal configurations $A$ for which all solutions are algebraic ([Beukers '11]), certainly for non-integer parameter vectors $\beta$. New techniques are needed.
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The End

Thank you for your attention!