A Combinatorial Formula for Idempotents in the 0-Hecke Algebra of the Symmetric Group

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Abstract.

Building on the work of P.N. Norton, we give combinatorial formulae for two maximal decompositions of the identity into orthogonal idempotents in the 0-Hecke algebra of the symmetric group, $\mathbb{C}H_0(S_N)$. This construction is compatible with the branching from $H_0(S_{N-1})$ to $H_0(S_N)$.

Goal

We identify a formula for two different maximal orthogonal decompositions of the identity into idempotents for the 0-Hecke algebra of the symmetric group. These decompositions obey a branching rule from $H_0(S_{N-1})$ to $H_0(S_N)$.

Definition (The 0-Hecke Monoid)

The 0-**Hecke monoid** $H_0(S_N)$ is generated by the collection π_i for i in the set $I = \{1, \dots, N-1\}$ with relations:

- Idempotence: $\pi_i^2 = \pi_i$,
- Commutation: $\pi_i\pi_j = \pi_j\pi_i$ for $|i-j| > 1$,
- Braid Relation: $\pi_i\pi_{i+1}\pi_i = \pi_{i+1}\pi_i\pi_{i+1}$.

The 0-**Hecke algebra** $\mathbb{C}H_0(S_N)$ is the monoid algebra of the 0-Hecke monoid. These relations are encoded in the *Dynkin diagram*.

The 0-Hecke monoid of the symmetric group can be considered as the monoid generated by the anti-sorting operators on permutations of N. These π_i act on permutation $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_N)$ by transposing σ_i and σ_{i+1} if $\sigma_i < \sigma_{i+1}$, and doing nothing otherwise.

Automorphisms

$\mathbb{C}H_0(S_N)$ is alternatively generated as an algebra by elements $\pi_i^- := (1-\pi_i)$, which satisfy the same relations as the π_i generators. There is a unique automorphism Ψ of $\mathbb{C}H_0(S_N)$ defined by $\pi_i \xrightarrow{\Psi} (1-\pi_i)$. For any longest element w_J^+, the image $\Psi(w_J^+)$ is a longest element in the $(1-\pi_i)$ generators; this element is denoted w_J^-.

$$\Psi : \pi_i \longrightarrow (1-\pi_i)$$
$$\Theta : \pi_i \longrightarrow \pi_{N-i}$$
$$\kappa : \pi_{i_1}\pi_{i_2}\cdots\pi_{i_{k-1}}\pi_{i_k} \longrightarrow \pi_{i_k}\pi_{i_{k-1}}\cdots\pi_{i_2}\pi_{i_1}$$

The idempotents constructed by the formula in this poster are fixed as a set under the automorphisms Ψ and Θ, but the automorphism κ can be used to obtain a second set of orthogonal idempotents.

Representation Theory

To each subset J of $I = \{1, 2, \dots, N-1\}$ is associated a simple and an indecomposable projective module. The simple modules are one dimensional, and are given by the map λ_J defined on the generators as follows:

$$\lambda_J(\pi_i) = \begin{cases} 0 & \text{if } i \in J, \\ 1 & \text{if } i \notin J. \end{cases}$$

The indecomposable (left) projective modules can be realized combinatorially as the collection of elements in the monoid whose right descent set is exactly J. Let w_J^+ be the longest element in the generators indexed by J, and $w_{\bar{J}}^-$ be $\Psi(w_{I\setminus J}^+)$. Then, by a theorem of P.N. Norton, the left projective module is:

$$H_0(S_N)w_J^+w_{\bar{J}}^-.$$

Unfortunately, these elements $w_J^+w_{\bar{J}}^-$ are neither orthogonal nor idempotent. The goal of this project was to obtain an orthogonal decomposition of the identity into idempotents.

Construction of Idempotents

- Choose a *signed diagram* D, by assigning a sign to each node of the Dynkin diagram. The 2^{N-1} signed diagrams are in bijection with the set of simple modules. For D a signed diagram, let D_i be the signed sub-diagram consisting of the first i entries of D.
- Construct the elements L_D and R_D, by taking the product of the longest elements in parabolic submonoids of adjacent matching signs, in the generators $\pi_i^\pm$ according to the sign in the diagram.
- Set $C_D = L_DR_D$. This is the *diagram demipotent* associated to D. We call an element x *demipotent* if there exists a finite positive integer m such that $x^m = x^m + 1$. (Thus, x^m is idempotent.)
- Finally, obtain the idempotent:

$$I_D = C_D^{N-1} = C_{D_1}C_{D_2}\cdots C_{D_N=D}.$$

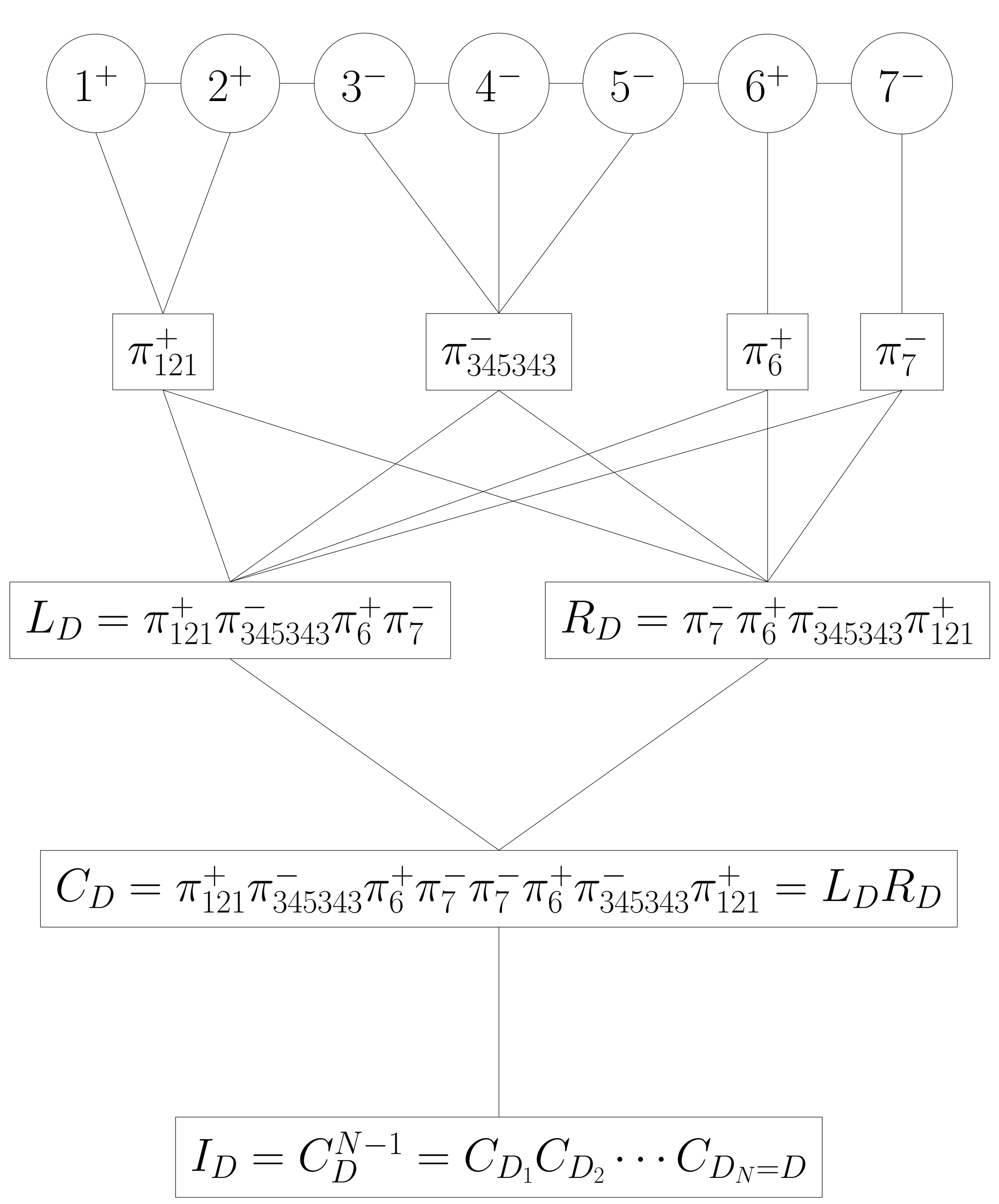

Figure: Construction of an Idempotent for $\mathbb{C}H_0(S_8)$.

Branching of Diagram Demipotents

Let $D+$ and $D-$ denote the signed diagrams equal to D with an extra $+$ or $-$ adjoined.

The diagram demipotents obey the branching rule from S_{N-1} to S_N:

$$C_D = C_{D+} + C_{D-}.$$

Furthermore, 'sibling' diagram demipotents $D+$ and $D-$ commute and are orthogonal:

$$C_{D-}C_{D+} = C_{D+}C_{D-} = 0.$$

We now state the main result.

Main Theorem

Each diagram demipotent C_D for $H_0(S_N)$ is demipotent, and yields an idempotent $I_D = C_{D_1}C_{D_2}\cdots C_D = C_D^N$. The collection of these idempotents $\{I_D\}$ form an orthogonal set of primitive idempotents that sum to 1.

Conjecture

Applying a 'mask' according to a signed Dynkin diagram D to the word

$$u_N = \pi_1\pi_2\cdots\pi_{N-1}\pi_N\pi_{N-1}\cdots\pi_2\pi_1$$

seems to produce demipotent elements which yield the same idempotents as the diagram demipotents C_D. This has been checked up to $N = 9$ using Sage-Combinat.

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