Barycentric subdivisions of simplicial complexes

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Setup

X: a d-dimensional simplicial complex

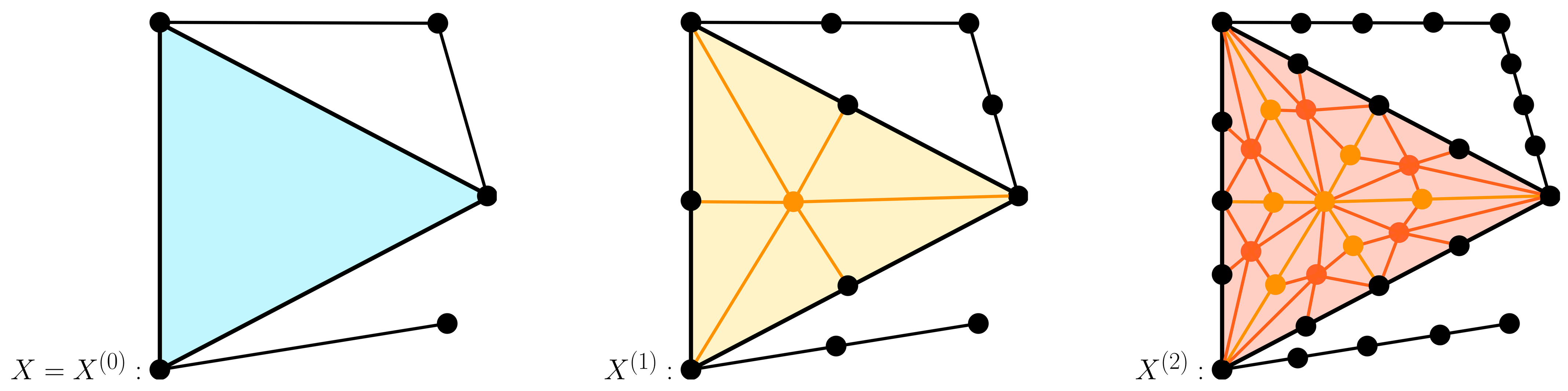
$X^{(k)}$: the k-fold barycentric subdivision of X

f_i^X: the number of i-faces of X

$f^X := (f_{-1}^X, f_0^X, \ldots, f_d^X)$: the f-vector of X

$f^X(t) := f_{-1}^X t^{d+1} + f_0^X t^d + \ldots + f_{d-1}^X t + f_d^X$: the f-polynomial of X

$q_n^X(t) := \frac{t^{d+1}}{[(d+1)!]^n} f^{X^{(n)}}(t^{-1})$



$f^X(t) = t^3 + 5t^2 + 6t + 1$ $\qquad$ $f^{X^{(1)}}(t) = t^3 + 12t^2 + 18t + 6$ $\qquad$ $f^{X^{(2)}}(t) = t^3 + 36t^2 + 72t + 36$

$q_0^X(t) = 1 + 5t + 6t^2 + t^3$ $\qquad$ $q_1^X(t) = \frac{1}{6} + 2t + 3t^2 + t^3$ $\qquad$ $q_2^X(t) = \frac{1}{36} + t + 2t^2 + t^3$ $\quad \ldots \longrightarrow q^2(t) = \frac{1}{2}t + \frac{3}{2}t^2 + t^3$

Main results

As n grows, the $q_n^X(t)$ converge to a limit polynomial $q^d(t)$ that does not depend on X but only on the dimension d. Its roots are contained in $[-1, 0]$ and are reflection-symmetric about $-\frac{1}{2}$.

The roots of the f-polynomials converge to the reciprocals of the roots of the $q^d(t)$.

Geometric intuition

As n grows, the contribution of the higher-dimensional cells to the number of new cells of dimensions i dominates the one of the lower-dimensional cells – for every i.

Motivation

Theorem 1 (Brenti and Welker, 2008 [1]). *As n grows, the roots of $f^{X^{(n)}}(t)$ converge towards d negative real numbers which depend only on d, not on X.*

Proof: Analytical methods. □

Question: Why? What are these numbers?

Geometric explanation

Let σ_i be the standard i-simplex, and let $f_j^{\mathring{\sigma}_i^{(1)}}$ denote the number of open j-cells in the interior of $\sigma_i^{(1)}$. Then

$$f^{X^{(n)}} = f^{X^{(n-1)}} F_d = f^X (F_d)^n$$

where $F_d := [f_j^{\mathring{\sigma}_i^{(1)}}]_{i,j=-1,\ldots,d}$ is lower triangular.

The matrix F_d has eigenvalues $0!,\ 1!, \ldots, d!$.

Let $\underline{t} := (t^{d+1}, \ldots, t^1, 1)$, $\overline{t} := (1, t, t^2, \ldots, t^{d+1})$. Then

$$f^{X^{(n)}}(t) = f^X P_d \operatorname{diag}(0!, 1!, \ldots, (d+1)!)^n P_d^{-1} \underline{t}$$

and thus

$$q_n^X(t) = f^X P_d \operatorname{diag}\left(\left[\frac{0!}{(d+1)!}\right]^n, \left[\frac{1!}{(d+1)!}\right]^n, \ldots, \left[\frac{(d+1)!}{(d+1)!}\right]^n \right) P_d^{-1} \overline{t}$$

$$\to f^X P_d \operatorname{diag}(0, \ldots, 0, 1) P_d^{-1} \overline{t}$$

Let then $\quad q^d(t) := \operatorname{diag}(0, \ldots, 0, 1) P_d^{-1} \overline{t}$

Notice: $q^d(t)$ clearly does not depend on X.

Since the $q_n^X(t)$ are monic and of same degree as $q^d(t)$, the roots of the $q_n^X(t)$ converge to the roots of $q^d(t)$ (see for instance [3]).

Hence for growing n the roots of $f^{X^{(n)}}(t)$ converge to the reciprocals of the roots of $q^d(t)$, that do not depend on X!

Subdivisions via formal power series

Theorem 2 (D., Pixton, Sabalka 2009).

$$B(e^{tx}) = \frac{1}{1 - (e^x - 1)t}$$

where $B : \mathbb{Z}[t][[x]] \to \mathbb{Z}[t][[x]]$ is given on monomials by the operator

$$\begin{aligned} b : \mathbb{Z}[t] &\to \mathbb{Z}[t] \\ g(t) &\mapsto \sum_{k \geq 0} \Delta^k \{g(l)\}_l t^k \end{aligned}$$

In fact, b represents barycentric subdivision because, for all j,

$$b(q_j^X(t)) = d! q_{j+1}^X(t).$$

Corollary 3.

$$b(q^d(t)) = d! q^d(t)$$

and since the eigenvalues of F_d are distinct, $q^d(t)$ is determined by this equation up to a sign.

Symmetry of the roots

Theorem 4 (D.,Pixton, Sabalka 2009).

$$b\iota = \iota b$$

as operators on $\mathbb{Z}[t]$, where

$$\begin{aligned} \iota : \mathbb{Z}[t] &\to \mathbb{Z}[t] \\ g(t) &\mapsto g(-1-t) \end{aligned}$$

Proof: The map ι is clearly an involution, and $B\iota(e^{tx}) = \iota B(e^{tx})$. □

With Corollary 3, it follows that

$$q^d(t) = (-1)^d q^d(-1-t).$$

Hence, the roots of $q^d(t)$ are distinct, contained in the interval $[-1, 0]$ and symmetric under reflection about $-\frac{1}{2}$. In particular, $-\frac{1}{2}$ is a root if and only if d is even.

Notice: Our method, via the operators b and ι, can be applied for more general subdivision methods and yields a limit polynomial with "symmetric" limit roots for this more general case as well. See [2].

Further questions

- Compute the coefficients of $q^d(t)$ "explicitly" from geometric properties of $\mathbb{R}^d$.
- Prove "combinatorially" that the roots of $q^d(t)$ are in $[-1, 0]$.
- What is the significance of the roots of $q^d(t)$? Do they have any topological or geometric meaning? In particular:
- Is the middle root $-\frac{1}{2}$ connected to the Euler Characteristic of the d-sphere?
- (asked by Lou Billera) Does the symmetry come from the Dehn-Sommerville equations?

Computations

Our methods yield explicit formulas for the coefficients of P_d, P_d^{-1}, and hence for the coefficients of $q^d(t)$. We list them up to $d = 9$.

t^1	t^2	t^3	t^4	t^5	t^6	t^7	t^8	t^9	t^{10}
1	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0
$\frac{1}{2}$	$\frac{3}{2}$	1	0	0	0	0	0	0	0
$\frac{2}{11}$	$\frac{13}{11}$	2	1	0	0	0	0	0	0
$\frac{1}{19}$	$\frac{25}{38}$	$\frac{40}{19}$	$\frac{5}{2}$	1	0	0	0	0	0
$\frac{132}{10411}$	$\frac{3004}{10411}$	$\frac{45}{29}$	$\frac{95}{29}$	3	1	0	0	0	0
$\frac{90}{34399}$	$\frac{3626}{34399}$	$\frac{61607}{68798}$	$\frac{245}{82}$	$\frac{385}{82}$	$\frac{7}{2}$	1	0	0	0
$\frac{15984}{33846961}$	$\frac{12351860}{372316571}$	$\frac{7924}{18469}$	$\frac{39221}{18469}$	$\frac{56}{11}$	$\frac{70}{11}$	4	1	0	0
$\frac{983304}{12980789207}$	$\frac{119432466}{12980789207}$	$\frac{2296176994}{12980789207}$	$\frac{536193}{429266}$	$\frac{919821}{214633}$	$\frac{567}{71}$	$\frac{588}{71}$	$\frac{9}{2}$	1	0
$\frac{1345248918720}{1230314327847308 71}$	$\frac{281136722386176}{123031432784730871}$	$\frac{4358731100}{67808366729}$	$\frac{42780833020}{67808366729}$	$\frac{1335075}{448471}$	$\frac{3478503}{448471}$	$\frac{1050}{89}$	$\frac{930}{89}$	5	1

The corresponding roots can be approximated as follows:

-1	0								
-1	$-.5$	0							
-1	$-.76112$	$-.23888$	0						
-1	$-.88044$	$-.5$	$-.11956$	0					
-1	$-.93787$	$-.68002$	$-.31998$	$-.06213$	0				
-1	$-.9668$	$-.79492$	$-.5$	$-.20508$	$-.0332$	0			
-1	$-.98189$	$-.86737$	$-.63852$	$-.36148$	$-.13263$	$-.01811$	0		
-1	$-.98996$	$-.91332$	$-.73961$	$-.5$	$-.26039$	$-.08668$	$-.01004$	0	
-1	$-.99437$	$-.94277$	$-.81205$	$-.61285$	$-.38715$	$-.18795$	$-.05723$	$-.00563$	0

References

[1] F. Brenti, V. Welker. f-vectors of barycentric subdivisions *Math. Z.* **259** n. 4 (2008), 849–865.

[2] E. Delucchi, A. Pixton, L. Sabalka. f-vectors of subdivided simplicial complexes. arXiv:1002.3201, 13 pp., 2009.

[3] E. E. Tyrtyshnikov. A brief introduction to numerical analysis. Birkhäuser, Boston 1997.