Pattern avoidance in partial permutations

Anders Claesson

Vít Jelínek

Eva Jelínková

Sergey Kitaev

Reykjavik University

Abstract

Motivated by the concept of partial words, we introduce an analogous concept of partial permutations.

We introduce pattern-avoidance in partial permutations and prove that most known results on Wilf equivalence of permutation patterns can be extended to partial permutations with an arbitrary number of holes.

We also show that Baxter permutations of a given length k correspond to a Wilf-type equivalence class with respect to partial permutations with $(k-2)$ holes. Lastly, we enumerate the partial permutations of length n with k holes avoiding a given pattern of length at most 4, for each $n \geq k \geq 1$.

Partial permutations

A **partial permutation of length n with k holes** is a word $\pi = a_1 a_2 \ldots a_n$ in which each of the elements $[n-k] = \{1, 2, \ldots, n-k\}$ appears exactly once; the remaining k symbols of π are "holes".

For example, $3 \diamond\diamond 4 \diamond 12$ is a length 7 partial permutation of $\{1, 2, 3, 4\}$ with 3 holes.

Let $\mathcal{S}_n^k$ = partial permutations of $[n-k]$ with exactly k holes.

$$\begin{aligned}
\mathcal{S}_3^0 &= \{123, 132, 213, 231, 312, 321\} \\
\mathcal{S}_3^1 &= \{12\diamond, 1\diamond 2, 21\diamond, 2\diamond 1, \diamond 12, \diamond 21\} \\
\mathcal{S}_3^2 &= \{1\diamond\diamond, \diamond 1\diamond, \diamond\diamond 1\} \\
\mathcal{S}_3^3 &= \{\diamond\diamond\diamond\}
\end{aligned}$$

Note that

$$|\mathcal{S}_n^k| = \binom{n}{k}(n-k)! = n!/k!$$

Note also that $\mathcal{S}_n^0$ is the familiar symmetric group $\mathcal{S}_n$.

For $H \subseteq [n]$, let

$$\mathcal{S}_n^H = \{\, a_1 \ldots a_n \in \mathcal{S}_n^{|H|} \,:\, a_i = \diamond \iff i \in H \,\}.$$

Pattern avoidance in partial permutations

Let $i_1 < \cdots < i_{n-k}$ be the indices of the non-hole elements of $\pi \in \mathcal{S}_n^k$. A permutation $\sigma \in \mathcal{S}_n$ is an **extension** of π if

$$\sigma(i_1)\ldots\sigma(i_{n-k}) \quad \text{is order isomorphic to} \quad \pi(i_1)\ldots\pi(i_{n-k}).$$

E.g., $2\diamond 1$ has three extensions: 312, 321 and 231.

$\pi \in \mathcal{S}_n^k$ **avoids** $p \in \mathcal{S}_\ell$ if each extension of π avoids p

E.g., $\pi = 32\diamond 154$ avoids 1234, but π does not avoid 123: the permutation 325164 is an extension of π and it contains two occurrences of 123.

$$\mathcal{S}_n^k(p) = \text{set of } p\text{-avoiding partial permutations in } \mathcal{S}_n^k$$
$$\mathcal{S}_n^H(p) = \text{set of } p\text{-avoiding partial permutations in } \mathcal{S}_n^H$$

k-Wilf-equivalence and $\star$-Wilf-equivalence

Two patterns p and q are

- **k-Wilf-equivalent** if $|\mathcal{S}_n^k(p)| = |\mathcal{S}_n^k(q)|$ for all n.
  (0-Wilf-equivalence is standard Wilf-equivalence)
- **$\star$-Wilf-equivalent** if p and q are k-Wilf-equivalent for all k.
- **strongly k-Wilf-equivalent** if $|\mathcal{S}_n^H(p)| = |\mathcal{S}_n^H(q)|$ for each n and for each k-element subset $H \subseteq [n]$.
- **strongly $\star$-Wilf-equivalent** if they are strongly k-Wilf-equivalent for all k.

Note that

- strong k-Wilf-equivalence $\implies$ k-Wilf-equivalence
- strong $\star$-Wilf-equivalence $\implies$ $\star$-Wilf-equivalence

The converse implications are not true.
Consider $p = 1342$ and $q = 2431$.

A partial permutation avoids p iff its reverse avoids q, and thus p and q are $\star$-Wilf-equivalent.

However, p and q are not strongly 1-Wilf-equivalent, and hence not strongly $\star$-Wilf-equivalent either.

To see this: let $H = \{2\}$; check $|\mathcal{S}_5^H(p)| = 13 \neq |\mathcal{S}_5^H(q)| = 14$.

We show that most pairs of Wilf-equivalent patterns that were discovered so far are in fact $\star$-Wilf-equivalent.

The only exception is the pair $p = 2413$ and $q = 1342$. These patterns are Wilf-equivalent [4], but they are neither 1-Wilf-equivalent nor 2-Wilf-equivalent.

Strong $\star$-Wilf-equivalence of $1\,2\ldots\ell\,\sigma$ and $\ell\ldots2\,1\,\sigma$

Theorem

For any $\ell \leq m$, and any permutation σ of $\{\ell+1, \ell+2, \ldots, m\}$, $1\,2\ldots(\ell-1)\,\ell\,\sigma$ is strongly $\star$-Wilf-equivalent to $\ell\,(\ell-1)\cdots2\,1\,\sigma$.

This is a strengthening of a result of Backelin, West and Xin [1], who show that patterns of this form are Wilf-equivalent.

Our proof is based on a different argument than theirs. The main ingredient of our proof is an involution on a set of fillings of Ferrers diagrams, discovered by Krattenthaler [3].

Strong $\star$-Wilf-equivalence of $3\,1\,2\,\sigma$ and $2\,3\,1\,\sigma$

Theorem

For any permutation σ of the set $\{4, 5, \ldots, m\}$, the two permutations 312σ and 231σ are strongly $\star$-Wilf-equivalent.

This generalizes a result of Stankova and West [5], who have shown that 312 and 231 are shape-Wilf-equivalent.

The original proof of Stankova and West [5] is rather complicated, and does not seem to admit a straightforward generalization to the setting of shape-$\star$-Wilf-equivalence.

Our proof is different and it is based on a bijection of Jelínek [2], obtained in the context of pattern-avoiding ordered matchings.

A new characterization of Baxter permutations

(k-Wilf equivalence of patterns whose length is small in terms of k)

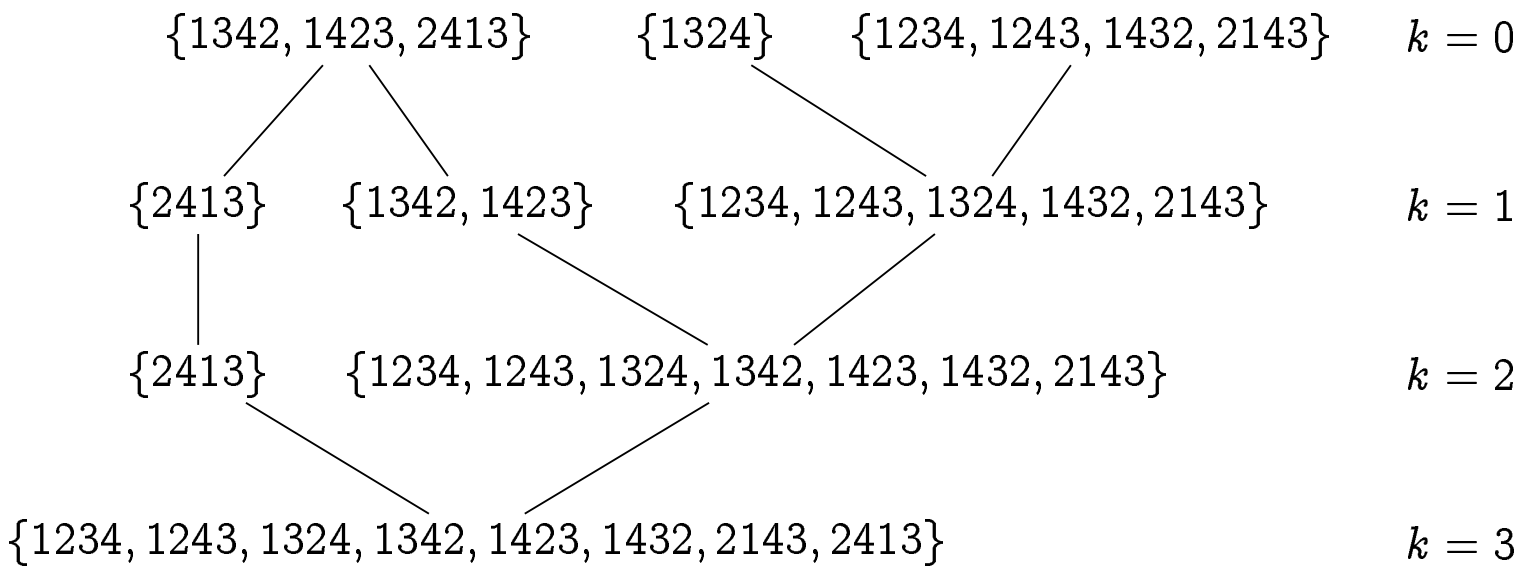
Theorem

Let $p \in \mathcal{S}_\ell$ be a permutation pattern. Let $k = \ell - 2$.

- *p is Baxter $\implies |\mathcal{S}_n^k(p)| = \binom{n}{k}$ for each $n \geq k$.*
- *p is not Baxter $\implies |\mathcal{S}_n^k(p)| < \binom{n}{k}$ for each $n \geq k + 3$.*

Moreover, all Baxter permutations of length ℓ are strongly $(\ell - 2)$-Wilf equivalent.

k-Wilf-equivalence for patterns of length 4



In this figure we depict the k-Wilf equivalence classes. Since all the k-Wilf equivalences are closed under complement and reversal (but not inversion), we represent the 24 patterns of length four by eight representatives, one from each symmetry class.

For instance, $\{1342, 1423\}$ in the second row represents the union of $\{1342, 2431, 3124, 4213\}$ and $\{1423, 2314, 3241, 4132\}$.

Enumeration for patterns of length 4

Theorem

The number of partial permutations of length $n \geq 1$ with a single hole, avoiding a pattern of length 4, is given here:

- $|\mathcal{S}_n^1(1234)| = |\mathcal{S}_n^1(1243)| =$
  $|\mathcal{S}_n^1(1324)| = |\mathcal{S}_n^1(1432)| = |\mathcal{S}_n^1(2143)| = \binom{2n-2}{n-1}$
- $|\mathcal{S}_n^1(1342)| = |\mathcal{S}_n^1(1423)| = \binom{2n-2}{n-1} - \binom{2n-2}{n-5}$
- $|\mathcal{S}_n^1(2413)| = \frac{2}{n+1}\binom{2n}{n} - 2^{n-1}$

An open problem

Easy

The $(\ell - 1)$-Wilf-equivalence class of $\mathrm{id}_\ell = 1\,2\ldots\ell$ contains every permutation of length ℓ.

We have shown

The $(\ell - 2)$-Wilf-equivalence class of id_ℓ contains exactly the Baxter permutations of length ℓ.

Open

What is the $(\ell - 3)$-Wilf-equivalence class of id_ℓ?

For $\ell \leq 4$ that class contains exactly the **layered permutations** of length ℓ. Computer enumeration suggests that the same is true for $\ell = 5$.

We do not know whether this holds for larger values of ℓ.

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