Wall Crossings for Double Hurwitz Numbers

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This is joint work with Paul Johnson (Imperial College) and Hannah Markwig (Goettingen).

Double Hurwitz Numbers (combinatorics)

The *Double Hurwitz number* $H_g^r(\alpha, -\beta)$:

"number" of $\sigma_0, \tau_1, \ldots, \tau_r, \sigma_\infty \in S_d$ such that:

- σ_0 has cycle type α;
- τ_i's are simple transpositions;
- σ_∞ has cycle type β;
- $\sigma_0 \tau_1 \ldots \tau_r \sigma_\infty = 1$
- the subgroup generated by such elements acts transitively on the set $\{1, \ldots, d\}$

The above number is multiplied by the automorphisms of the permutations σ_0, σ_∞ and divided by $d!$.

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Paul's insight

One can count Hurwitz numbers by counting movies of the monodromy representation, i.e. organizing the count by the cycle types of the successive products:

$$\sigma_0, \sigma_0\tau_1, \sigma_0\tau_1\tau_2 \dots$$

Via the cut and join equations, such movies can be represented by trivalent graphs. Hence:

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The formula

Theorem (C, Johnson, Markwig, 2008)

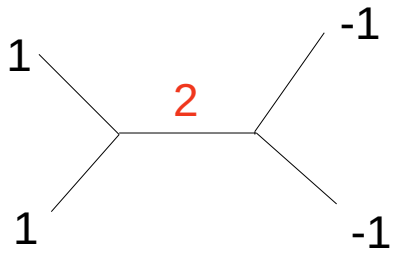
$$H_g^r(\alpha, -\beta) = Aut(\alpha)Aut(\beta) \sum_{\Gamma} \frac{1}{Aut(\Gamma)} \prod_{IE} e_i \tag{1}$$

- Γ are trivalent, oriented, genus g graphs.
- the edges have positive integer weights satisfying the 0-tension condition at each vertex.
- the ends are labelled by the parts of α and β.
- the vertices are totally ordered (compatibly with the edges).
- IE stands for “internal edges".

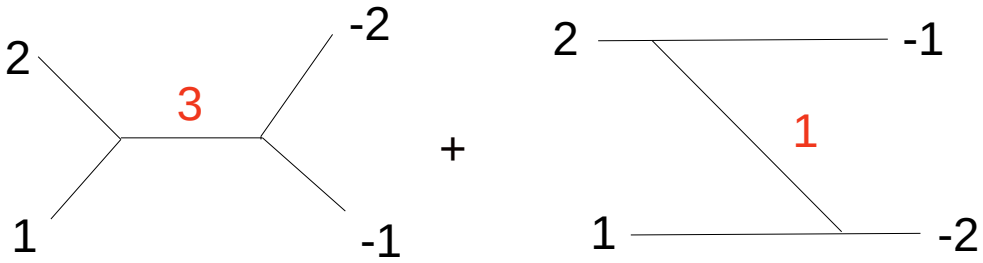
Examples

Some silly examples:

$$H_0^2((1,1),-(1,1)) = 2$$



$$H_0^2((2,1),-(2,1)) = 4$$



Structure of double Hurwitz Numbers

Think of double Hurwitz numbers as functions in the entries of the partitions:

$$\begin{array}{rccc} H_g^r(-): & \mathcal{H} \subseteq \mathbb{R}^{\ell(\alpha)+\ell(\beta)} & \to & \mathbb{R} \\ & (\alpha_1, \ldots, -\beta_{\ell(\beta)}) & \mapsto & H_g^r(\alpha, -\beta) \end{array}$$

where

$$\mathcal{H} = \left\{ \sum_{i=1}^{\ell(\alpha)+\ell(\beta)} x_i = 0 \right\}$$

Goulden, Jackson, Vakil:

1. $\mathcal{H}$ is subdivided into a finite number of chambers, inside each of which the Hurwitz numbers are polynomials in the entries of the partitions.
2. polynomials have nonzero coefficients between top degree $4g - 3 + \ell(\alpha) + \ell(\beta)$ and bottom degree $2g - 3 + \ell(\alpha) + \ell(\beta)$.
3. polynomials are either even or odd (according to parity of leading degree).
4. polynomials should be interpreted as intersection numbers on some moduli space.

The genus 0 story

In genus 0, Shapiro, Shadrin, Vainshtein settle the story:

1. They describe the location of the walls.
2. They give a closed formula for the Hurwitz number in one chamber.
3. They describe wall crossing formulas at any wall. It has the flavor of a degeneration formula.

Specifically:

Wall: $\sum_I x_i = 0$.

Set $\delta = \sum_I x_i$.

Side 2:= $\{\sum_I x_i > 0\}$
Side 1:= $\{\sum_I x_i < 0\}$

Wall Crossing Formula:

$$\begin{aligned} WC(x) &= P_2(x) - P_1(x) \\ &= \binom{r}{r_1, r_2} \delta H_0^{r_1}(x_I, -\delta) H_0^{r_2}(x_{I^c}, \delta) \end{aligned}$$

Main Result

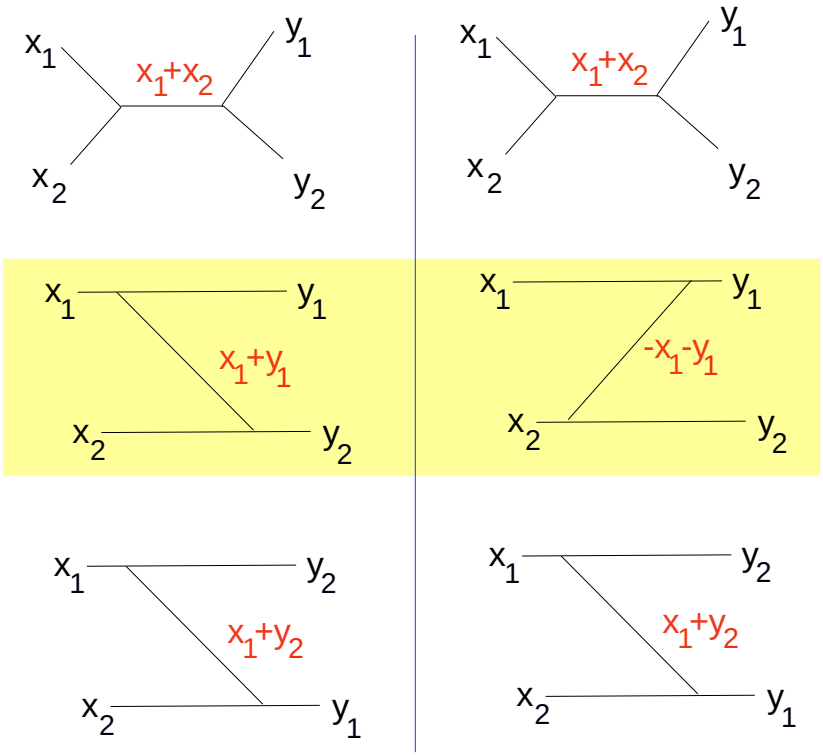
Theorem (C, Johnson, Markwig, 2009)

Location of walls and degeneration-type wall crossing formulas are described in arbitrary genus g.

Example

$x_1 + y_2 > 0$

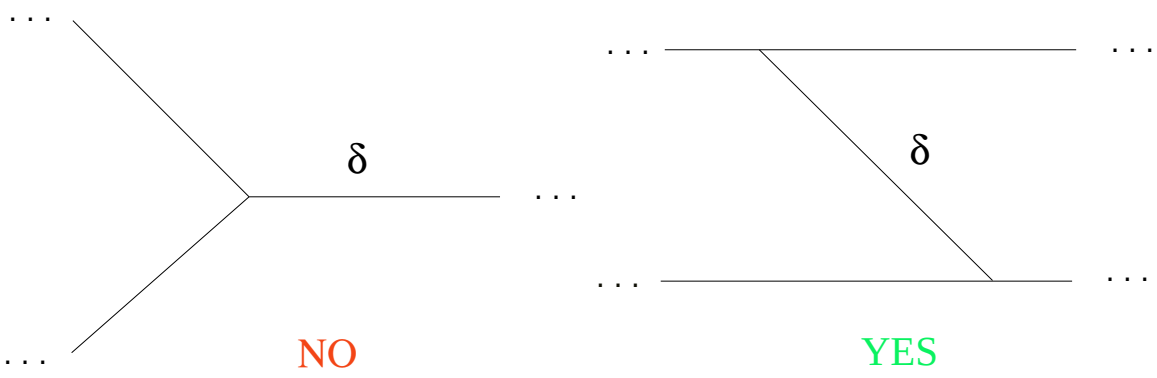
$x_1 + y_1 > 0$ $\quad$ $x_1 + y_1 = 0$ $\quad$ $x_1 + y_1 < 0$



$2x_1$ $\quad$ $2(x_1+y_1)$ $\quad$ $-2y_1$

Sketch of proof in $g = 0$

- The graphs that can contribute to the wall crossing must contain an edge labelled δ.
- Further, this edge must be allowed to “flip".



- When cutting along δ, note that the graph of each connected component is a graph used to compute the Hurwitz numbers corresponding to the splitting of inputs and outputs prescribed by the wall.

Sketch of proof in $g = 0$

- Conversely, start from two graphs used to compute the Hurwitz numbers for the connected components. Each such graph comes with a total order of the vertices, and there are $\binom{r}{r_1, r_2}$ ways to merge the orders on each components to a total order of ALL vertices. Gluing the graphs along δ, you obtain a graph contributing to the wall crossing!
- The contributions:

WC:

$$\prod_{IE \in \Gamma} e_k = \delta \prod_{IE \in \Gamma_1} e_i \prod_{IE \in \Gamma_2} e_j$$

H:

$$\prod_{IE \in \Gamma_1} e_i \prod_{IE \in \Gamma_2} e_j$$

Remark

What makes our proof nice and easy is that we have a geometric bijection between the graphs contributing to both sides of the formula. The polynomial contributions of each graph then just follow along for a ride!

$$Cut : \Gamma \leftrightarrow (\Gamma_1, \Gamma_2, m(\Gamma_1, \Gamma_2)) : Glue$$

The woes of higher genus:

1. There is a g dimensional polytope parameterizing admissible flows on a graph with assigned ends. The polynomial contribution to the Hurwitz numbers is obtained by summing the product of the edges over all integral points of this polytope.
2. There is NOT a natural bijection between the graphs appearing on two sides of the formula. A very subtle inclusion/exclusion is required.

The precise statement of the theorem

Theorem (C, Johnson, Markwig, 2009)

For the wall $\delta = \sum_I x_i = 0$

$$WC(x) = \sum_{\substack{r_1 + r_2 + r_3 = r \\ |\lambda| = |\eta| = \delta}} (-1)^{r_2} \binom{r}{r_1, r_2, r_3} \frac{\prod \lambda_i}{\ell(\lambda)!} \frac{\prod \eta_j}{\ell(\eta)!} H^{r_1}(x_I, -\lambda) H^{r_2 \bullet}(\lambda, -\eta) H^{r_3}(\eta, x_{I^C})$$

Computing the Hurwitz number:

For a graph Γ with assigned end weights x_i:

1. The bounded chambers of the cographic arrangement correspond to orientations of Γ.
2. Integral lattice points in each chamber parameterize admissible flows.
3. Contribution to the Hurwitz number is a sum of a polynomial φ over these lattice points. For each chamber, the result is a polynomial in the bounds of the polytope (which are linear polynomials in the x_i).
4. the polynomial φ is essentially always the same. For each chamber A it gets a signed multiplicity $m(A)$.

Crossing a Wall:

1. Varying the x_i translates the hyperplanes of the cographic arrangement.
2. A wall occurs when the topology of the arrangement changes, i.e. when a collection of hyperplanes intersects more non-transversely.
3. Chambers of the cographic arrangement form a basis for g-th relative homology group. Hence we have a H_g bundle over our parameter space for Hurwitz numbers.
4. The Gauss-Manin connection ∇ tells us how to relate chambers on either side of the wall.
5. The wall crossing contribution is obtained by summing φ over:

$$m(A_i^2)A_i^2 - m(B_j^1)\nabla(B_j^1)$$

Key Ingredient in Proof

We give a combinatorial formula for the Gauss Manin connection in terms of (an inclusion-exclusion of) cutting and regluing of graphs.

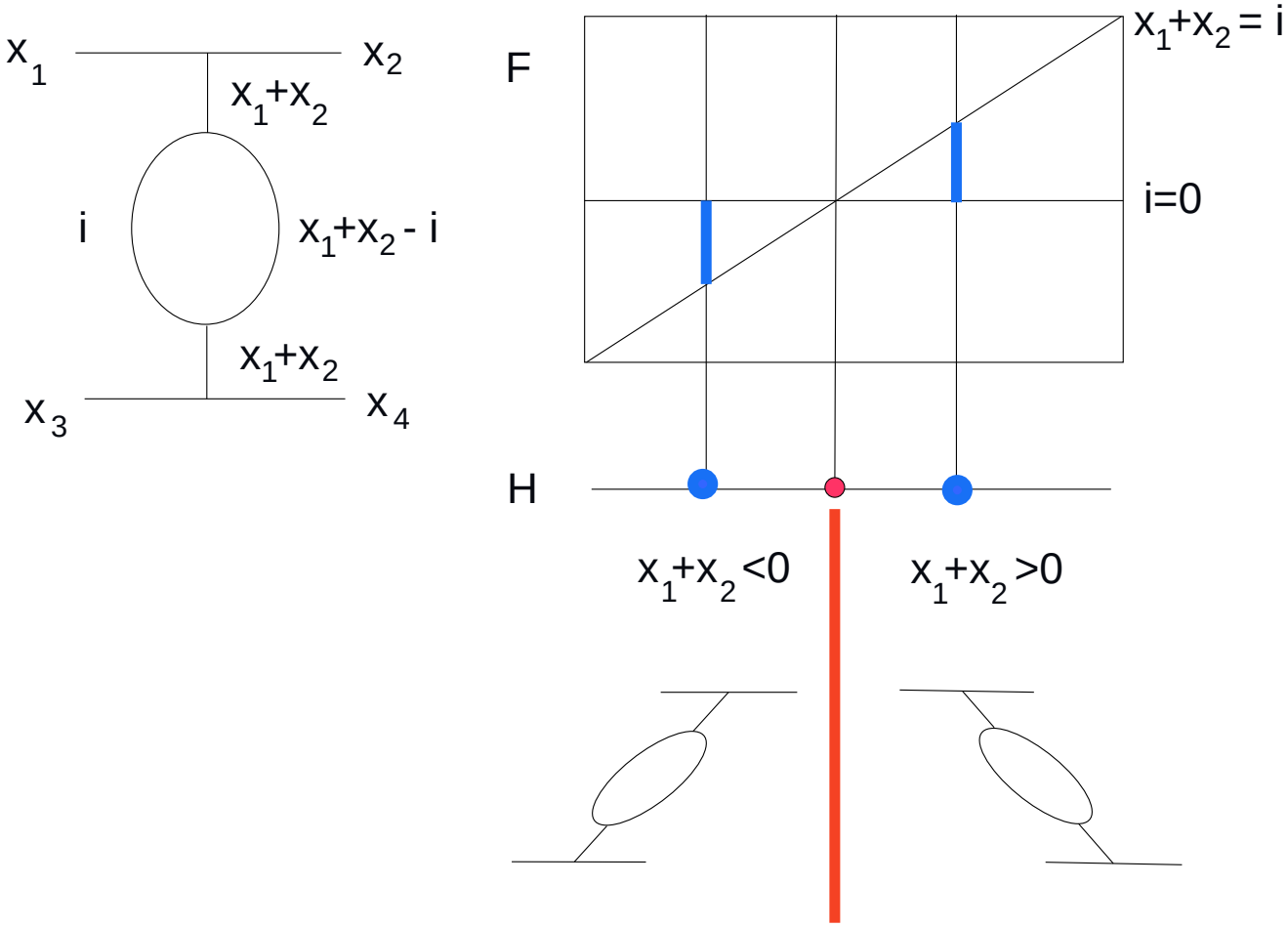
The combinatorial Gauss Manin connection

Applying the Gauss Manin connection to a chamber corresponding to a directed graph Γ:

1. Define a set of “cuttable edges".
2. Must cut cuttable edges in all possible legal ways.
3. Only cut edges are allowed to flop when reglued.
4. Direction of reglued edges is determined by merging of orders of vertices of components.
5. Inclusion/exclusion on number of connected components.

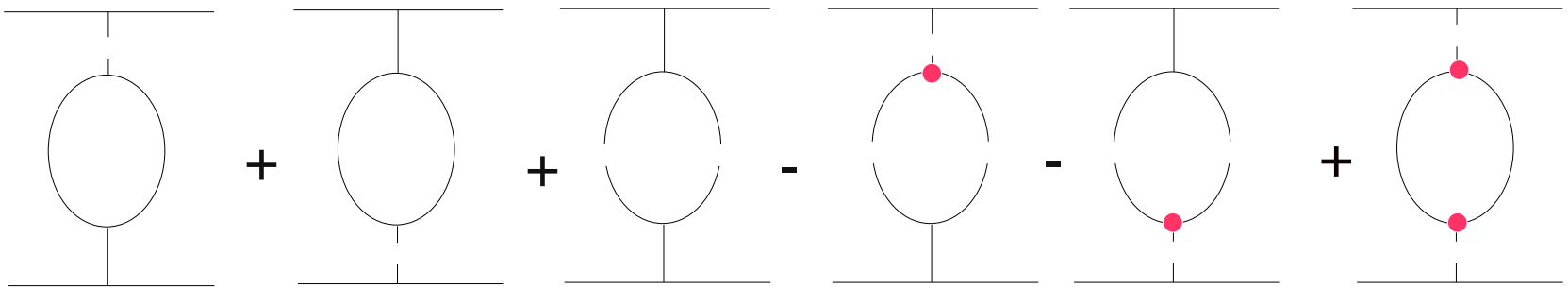
With this procedure, graphs contributing to the wall crossing are matched with graphs used to compute the Hurwitz numbers corresponding to the cuts of the graph, just like in genus 0.

An example



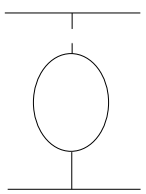
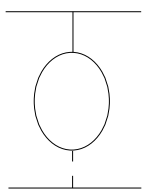
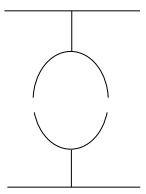
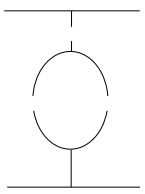
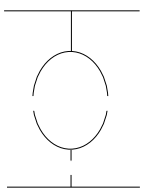
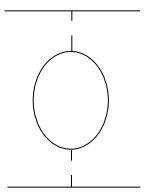
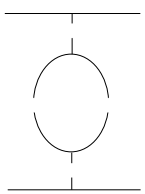
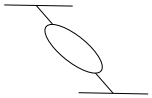
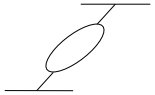
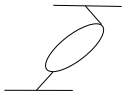
$$\frac{WC_\Gamma(x)}{(x_1+x_2)^2} = \sum_{0}^{x_1+x_2} i(x_1+x_2-i) - \sum_{x_1+x_2}^{0} i(x_1+x_2-i) = 2\sum_{0}^{x_1+x_2} i(x_1+x_2-i)$$

How to recover 2



$$\binom{4}{1} + \binom{4}{3} + \binom{4}{2} - \binom{4}{1,1,2} - \binom{4}{2,1,1} + \binom{4}{1,2,1} = 2$$

That 2 coming from combinatorial Gauss Manin

								
	+	+	+	-	-	-	+	1
							+	1
					-		+	0