Counting Polynomials as Hilbert Functions

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Flows and Tensions

modular tension polynomial
$\bar\theta_G(k)$ = #nowhere-zero $\mathbb{Z}_k$-tensions

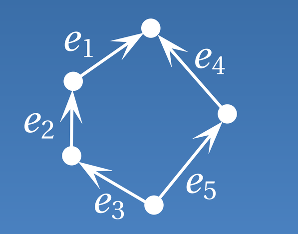
integral tension polynomial
$\theta_G(k)$ = #nowhere-zero k-tensions

modular flow polynomial
$\bar\varphi_G(k)$ = #nowhere-zero $\mathbb{Z}_k$-flows

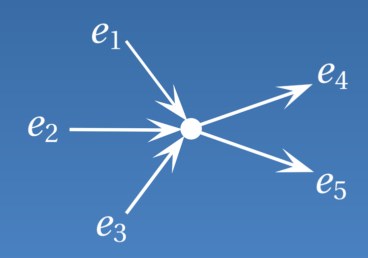
integral flow polynomial
$\varphi_G(k)$ = #nowhere-zero k-flows

$\mathbb{Z}_k$-**tension** $t : E \to \mathbb{Z}_k$
k-**tension** $t : E \to \{-k+1, \dots, k-1\}$
such that along every cycle tension is conserved

$\mathbb{Z}_k$-**flow** $f : E \to \mathbb{Z}_k$
k-**flow** $f : E \to \{-k+1, \dots, k-1\}$
such that at every vertex flow is conserved



$f(e_1)+f(e_2)+f(e_3) = f(e_4)+f(e_5)$

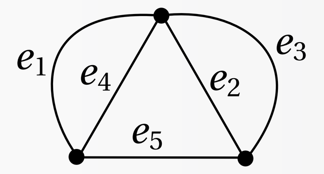


Motivation

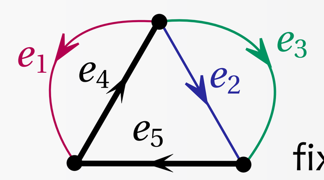
Theorem. *(Steingrímsson 2001)*
The chromatic polynomial $\chi_G(k+1)$ of a graph G is the Hilbert function of a relative Stanley-Reisner ideal.

Question
Do other counting polynomials in graph theory have the same property?

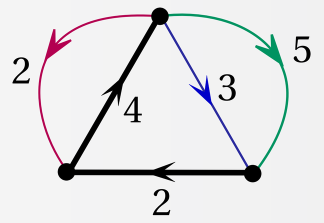
Example: $\mathbb{Z}_k$-flows



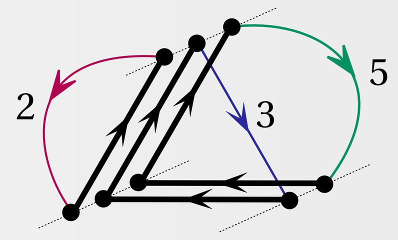
start with a graph G



fix an orientation and a spanning tree T (in black)



a $\mathbb{Z}_6$-flow on G

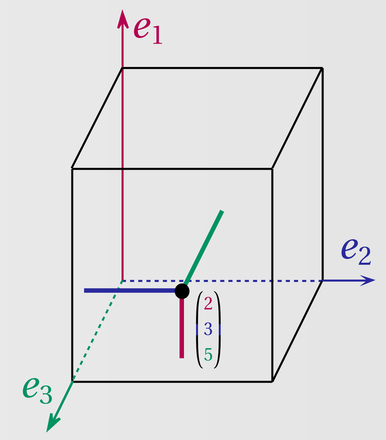


a flow is determined by its values on the non-tree edges

flow on non-tree edges

$$\begin{matrix} & e_1\ e_2\ e_3 \\ \begin{matrix} e_4 \\ e_5\end{matrix} & \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}\end{matrix} f|_{E\setminus T} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} = f|_T$$

network matrix — flow on tree edges



view $\mathbb{Z}_6$-flow as a point $f \in \mathbb{Z}^{E\setminus T} \cap 6 \cdot [0,1)^{E\setminus T}$

nowhere-zero $\mathbb{Z}_6$-flow lies off hyperplanes in $6 \cdot \mathscr{H}$

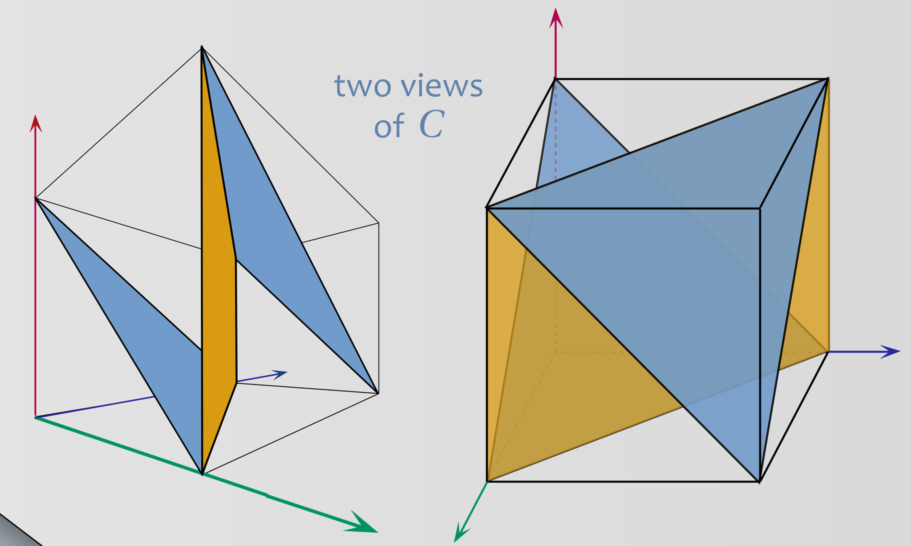
$$\mathscr{H} = \left\{ \begin{array}{ll} f_1+f_2+f_3 = l, & \\ f_2+f_3 = l, & l \in \mathbb{Z} \\ f_i = l, & i \in \{1,2,3\} \end{array} \right\}$$

first two determined by network matrix — determined by cube

relative polytopal complex

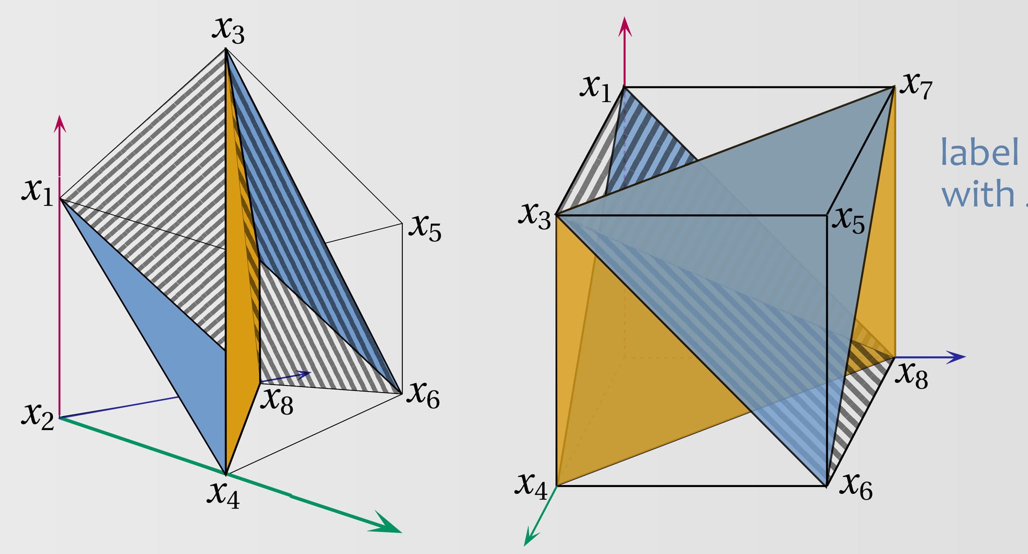
$C' \subset C$

network matrix hyperplanes & boundary of the cube — subdivision of the cube



C is integral and compressed

A pulling triangulation $\Delta' \subset \Delta$ of $C' \subset C$

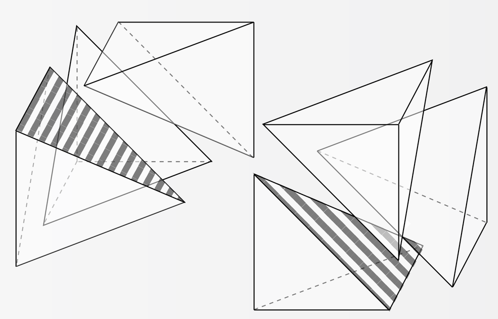


striped triangles introduced by triangulation, not contained in Δ'

$$\bar\varphi_G(k) = \mathrm{H}_{I_{\Delta/\Delta'}}(k) = (k-1)(k-2)^2$$

$$= 4\binom{k-1}{3} + 2\binom{k}{3} = 6\binom{k-1}{3} + 2\binom{k-1}{2}$$

#tetrahedra with 4 facets removed; #tetrahedra with 3 facets removed; #open tetrahedra; #open triangles



exploded view of Δ

$$I_\Delta = \langle \underbrace{x_1x_6, x_2x_5, x_4x_7}_{\text{cube diagonals}}, \underbrace{x_1x_5, x_2x_3, x_2x_7, x_2x_6, x_4x_5, x_5x_8}_{\text{facet diagonals}} \rangle$$

$$I_{\Delta/\Delta'} = \langle \underbrace{x_1x_3x_8, x_3x_6x_8}_{\text{triangles in } \Delta\setminus\Delta'}, \underbrace{x_1x_2x_4x_8, x_3x_5x_6x_7}_{\subset\text{-minimal tetrahedra in } \Delta\setminus\Delta'} \rangle \subset \mathbb{K}[\Delta]$$

$$\mathbb{K}[\Delta] = \mathbb{K}[x_1,\dots,x_8]/I_\Delta$$

Theorem

Let G be a graph. Then

the modular and integral flow polynomials and the modular and integral tension polynomials of G

are Hilbert functions of relative Stanley-Reisner ideals.

Relative Polytopal Complexes

A d-dimensional polytope P is **integral** if all vertices of P have integer coordinates.
P is called **compressed** if every pulling triangulation of P is unimodular.
A **relative polytopal complex** is a pair $C' \subseteq C$ of polytopal complexes.

$\bigcup C \setminus \bigcup C'$ is the set of points $x \in \mathbb{R}^n$ contained in C but not in C', that is,

$$\bigcup C \setminus \bigcup C' = \bigcup_{P \in C} P \setminus \bigcup_{P' \in C'} P'.$$

For any set $X \subset \mathbb{R}^n$ the **Ehrhart function** $\mathrm{L}_X : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 0}$ is given by
$\mathrm{L}_X(k) := \#\mathbb{Z}^n \cap k \cdot X.$

Fact. If $C' \subseteq C$ is an integral relative polytopal complex, then $\mathrm{L}_{\bigcup C \setminus \bigcup C'}(k)$ is a polynomial in k called the **Ehrhart polynomial** of $C' \subseteq C$.

Relative Stanley-Reisner Ideals

Stanley-Reisner ideal
$I_\Delta := \langle x^u \mid \mathrm{supp}(u) \notin \Delta \rangle \subseteq \mathbb{K}[x_1,\dots,x_n]$

Stanley-Reisner ring
$\mathbb{K}[\Delta] := \mathbb{K}[x_1,\dots,x_n]/I_\Delta$

Relative Stanley-Reisner ideal
$I_{\Delta/\Delta'} := \langle x^u \mid \mathrm{supp}(u) \notin \Delta' \rangle \subseteq \mathbb{K}[\Delta]$

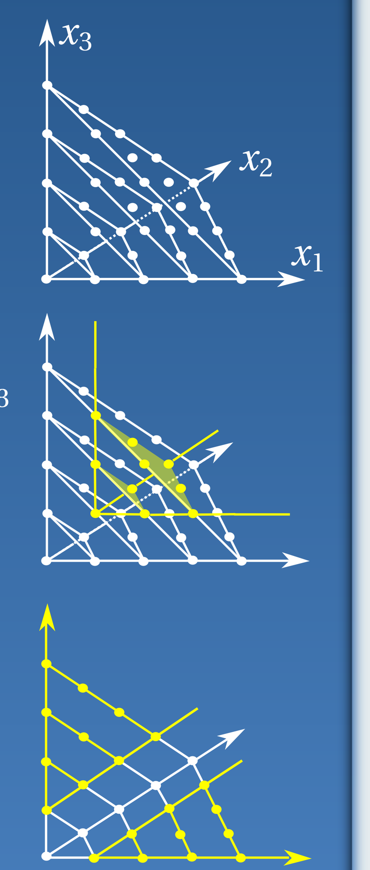
The **Hilbert function**
$\mathrm{H}_{I_{\Delta/\Delta'}}(k)$ counts monomials of degree k in $I_{\Delta/\Delta'} \subseteq \mathbb{K}[\Delta]$.

Example:

$\mathbb{K}[x] := \mathbb{K}[x_1, x_2, x_3]$

Δ = (x_1 — x_2 — x_3)
$I_\Delta = \langle x_1x_3 \rangle$

$\Delta' = x_2$
$I_{\Delta/\Delta'} = \langle x_1, x_3 \rangle \subset \mathbb{K}[x]/I_\Delta$



The Hilbert function of $I_{\Delta/\Delta'}$ is $H_{I_{\Delta/\Delta'}}(k) = 2k$.

Hilbert equals Ehrhart

Theorem.
Let C be a polytopal complex.

If all faces of C are compressed lattice polytopes, then
for any subcomplex $C' \subset C$
there exists a relative Stanley-Reisner ideal $I_{\Delta,\Delta'}$
such that for all $k \in \mathbb{Z}_{>0}$

$$\mathrm{L}_{\bigcup C \setminus \bigcup C'}(k) = \mathrm{H}_{I_{\Delta/\Delta'}}(k).$$

Ehrhart polynomial — Hilbert function

Bounds on the Coefficients

Theorem.
A polynomial $f(k) = \sum_{i=0}^{d} f_i \binom{k-1}{i}$ is the Hilbert function of some relative Stanley-Reisner ideal *if and only if*

$f_i \in \mathbb{Z}_{\geq 0}$ for all $0 \leq i \leq d$.

Better Bounds on the Coefficients

...exploiting the geometry of inside-out polytopes.
[See separate article arXiv:1004.3470.]

Theorem.
Let p denote the modular flow or tension polynomial of a graph.
Let $d+1 = \deg p$ and define the h-vector $(h_0, \dots, h_{d+1})$ of the polynomial $(k+1)^{d+1} - p(k)$ by

$$1 + \sum_{k \geq 1} \left((k+1)^{d+1} - p(k)\right) z^k = \frac{h_0 z^0 + \dots h_{d+1} z^{d+1}}{(1-z)^{d+1}}.$$

Then
1. $h_0 \leq h_1 \leq \dots \leq h_{\lfloor d/2 \rfloor}$,
2. $h_i \leq h_{d-i}$ for $i \leq d/2$,
3. $(h_0, h_1 - h_0, h_2 - h_1, \dots, h_{\lfloor d/2 \rfloor} - h_{\lfloor d/2 \rfloor - 1})$ is an M-vector.