A unified bijective method for maps: application to two classes with boundaries

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**Basic definitions.**
A planar map is a connected planar graph embedded in the plane. Our maps are simple: no loops nor multiple edges. A $d$-angulation is a map with faces of degree $d$. Triangulations and quadrangulations correspond to $d = 3$ and $d = 4$.

**Goals.**

Goal 1. Give a unified presentation of two existing bijections between maps and decorated plane trees ([6] for triangulations, and [10] for quadrangulations) by showing that both can be seen as a specialization of a "master bijection" $\Phi$.

Goal 2. Use a similar systematic approach to deal with triangulations and quadrangulations with a boundary.

**Master bijections $\Phi$, $\Phi_\star$.**

Orientations. An orientation is minimal if there is no counterclockwise directed cycles. An orientation is accessible from a vertex $v$ if there is a directed path from $v$ to any vertex. We denote by $O$ the set of minimal orientations which are accessible from the outer vertices and such that the outer-face is a simple directed cycle. We denote by $O'$ the subset of these orientations such that every edge from an outer vertex to an inner vertex is directed toward the inner vertex.

Mobiles. A mobile is a bicolored plane tree with some buds (half-edges) attached to black vertices. Its excess is the number of edges minus the number of buds.

The master bijections $\Phi_\star, \Phi$. Let $O$ be an orientation in $O$. The mobile $\Phi_\star(O)$ (resp. $\Phi(O)$) is defined by:

- black vertices ↔ inner faces of $O$.
- white vertices ↔ inner vertices of $O$.
- edges ↔ inner corners of $O$ preceding an incoming (inner) edge.
- buds ↔ other inner corners of $O$.

**Theorem.** $\Phi_\star$ is a bijection between oriented maps in $O$ and mobiles with positive excess. $\Phi$ is a bijection between oriented maps in $O$ and mobiles with negative excess.

**Triangulations/Quadrangulations without boundary.**

Orientations. The Euler relation implies that a triangulation (resp. quadrangulation) with $v$ inner vertices has $3v$ (resp. $2v$) inner edges. A $k$-orientation is an orientation such that every inner vertex has indegree $k$ and every outer vertex has indegree 1.

Proposition. By [9], a triangulation admits a $3$-orientation if and only if it is simple. By [7], a quadrangulation admits a $2$-orientation if and only if it is simple.

In this case there is a unique $k$-orientation in $O$.

**Theorem.** The master bijection $\Phi$ induces a bijection between simple triangulations with $v$ inner vertices and mobiles with $v$ white vertices such that:

- black vertices have degree 3.
- white vertices have degree 3.

**Theorem.** The master bijection $\Phi_\star$ induces a bijection between simple quadrangulations with $v$ inner vertices and mobiles with $v$ white vertices such that:

- black vertices have degree 4.
- white vertices have degree 2.

**Triangulations/Quadrangulations with boundary.**

A $p$-annular triangulation is a map with one inner simple face of degree $p$ and the other faces of degree 3.

Separation. A $p$-annular triangulation is separated if there is a cycle of length 3 separating the outer face and the $p$-gonal face. Any $p$-annular $d$-angulation $A$ decomposes into a 3-annular triangulation (i.e., triangulation with a marked inner triangle) and a non-separated $p$-annular triangulation.

**Orientation.** Let $p > 3$. Any $p$-annular triangulation $A$ admits a unique minimal orientation such that the $p$-gon is a directed cycle and any vertex not on the $p$-gon has indegree 1. This orientation is in $O$ if and only if $A$ is non-separated.

**Theorem.** The master bijection $\Phi_\star$ induces a bijection between simple non-separated $p$-annular triangulations and mobiles such that:

- black vertices have degree 3 except a special vertex of degree $p$ with no buds.
- white vertices have degree 3 except neighbors of the special vertex having degrees summing to $2p - 3$.

**Counting results (recovering Brown [4,5]).** Let $t_{n,p}$ be the number of simple $p$-gonal triangulations with $n + p$ vertices rooted in the $p$-gonal face. The formal series $T_p(x) = \sum_{n \geq 0} t_{n,p} x^n$ satisfies $T_p(x) := \left(\frac{1}{x^p} - \frac{1}{x^{p-1}}\right) x^{2p-3}$, where $u = 1 + x u^t$. Consequently, the Lagrange inversion formula gives: $t_{n,p} = \frac{1}{p^n} \left( n + p - 1 \right)! \left( \frac{p}{p-1} \right)!$.

The same strategy applies to quadrangulations (and pentagulations, etc. [†]).

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