A unified bijective method for maps: application to two classes with boundaries

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Basic definitions.

A *planar map* is a connected planar graph embedded in the plane.
Our maps are *simple*: no loops nor multiple edges.
A d-*angulation* is a map with faces of degree d.
Triangulations and *quadrangulations* correspond to $d = 3$ and $d = 4$.

Goals.

Goal 1. Give a unified presentation of two existing bijections between maps and decorated plane trees ([6] for triangulations, and [10] for quadrangulations) by showing that both can be seen as a specialization of a “master bijection” Φ.

Goal 2. Use a similar systematic approach to deal with triangulations and quadrangulations with a boundary.

Relation to previous and future works.

About Φ. The master bijection Φ is a slight extension of a bijection given in [1] (and in [2] for higher genus). The bijection [4] by Bouttier *et al.* is a specialization of Φ.
About triangulations/quadrangulations with a boundary. The counting results were given by Brown [4,5]. A bijection for triangulations with a boundary is given in [8], but the proofs use the results of Brown.
Future work. The bijections presented here have been extended in [*] to d-angulations for all d. In a future work [**], we extend our approach to count maps by degree and girth.

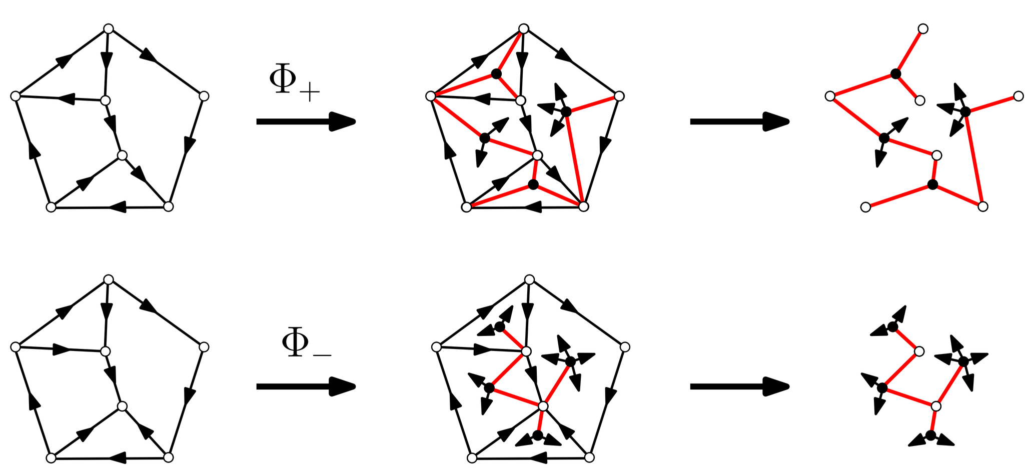
Master bijections Φ_-, Φ_+.

Orientations.
An orientation is *minimal* if there is no counterclockwise directed cycles.
An orientation is *accessible* from a vertex v if there is a directed path from v to any vertex.

We denote by $\mathcal{O}$ the set of minimal orientations which are accessible from the outer vertices and such that the outer-face is a simple directed cycle.
We denote by $\widetilde{\mathcal{O}}$ the subset of these orientations such that every edge from an outer vertex to an inner vertex is directed toward the inner vertex.

Mobiles. A *mobile* is a bicolored plane tree with some *buds* (half-edges) attached to black vertices. Its *excess* is the number of edges minus the number of buds.

The master bijections Φ_+, Φ_-.
Let O be an orientation in $\mathcal{O}$. The mobile $\Phi_+(O)$ (resp. $\Phi_-(O)$) is defined by:
- black vertices $\leftrightarrow$ inner faces of O.
- white vertices $\leftrightarrow$ (inner) vertices of O.
- edges $\leftrightarrow$ inner corners of O preceding an ingoing (inner) edge.
- buds $\leftrightarrow$ other inner corners of O.



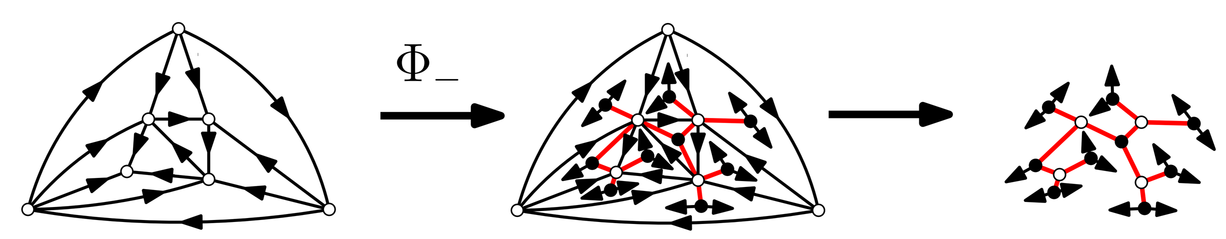
Theorem.
Φ_+ is a bijection between oriented maps in $\mathcal{O}$ and mobiles with positive excess.
Φ_- is a bijection between oriented maps in $\widetilde{\mathcal{O}}$ and mobiles with negative excess.

Triangulations/Quadrangulations without boundary.

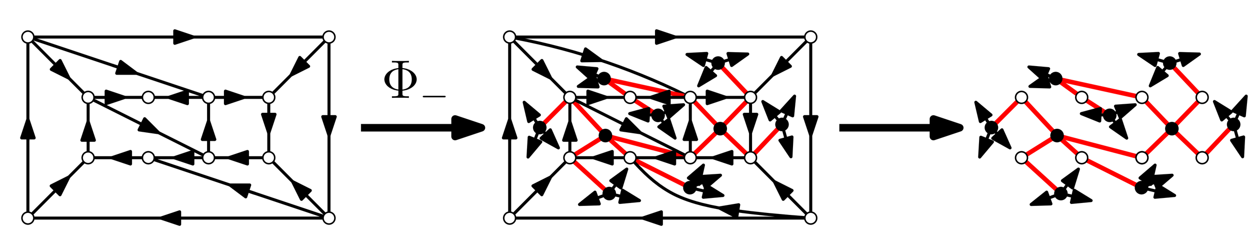
Orientations.
The Euler relation implies that a triangulation (resp. quadrangulation) with v inner vertices has $3v$ (resp. $2v$) inner edges.
A k-orientation is an orientation such that every inner vertex has indegree k and every outer vertex has indegree 1.

Proposition.
By [9], a triangulation admits a 3-orientation if and only if it is simple.
By [7], a quadrangulation admits a 2-orientation if and only if it is simple.
In this case there is a unique k-orientation in $\widetilde{\mathcal{O}}$.

Theorem. The master bijection Φ_- induces a bijection between simple triangulations with v inner vertices and mobiles with v white vertices such that
- black vertices have degree 3,
- white vertices have degree 3.



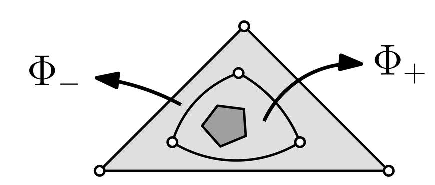
Theorem. The master bijection Φ_- induces a bijection between simple quadrangulations with v inner vertices and mobiles with v white vertices such that
- black vertices have degree 4,
- white vertices have degree 2.



Triangulations/Quadrangulations with boundary.

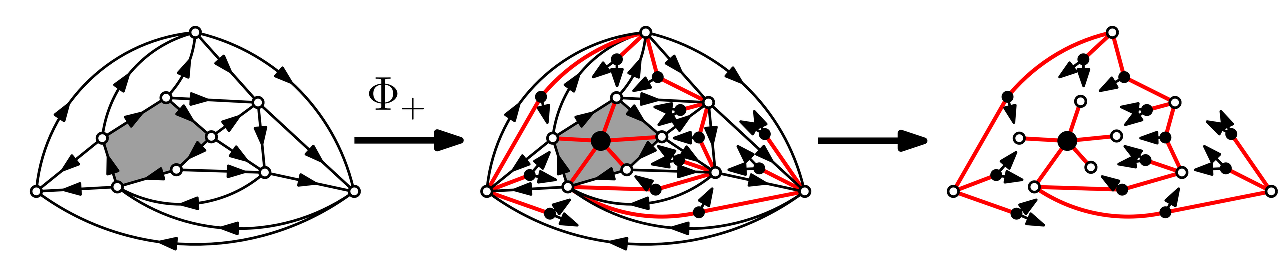
A p-*annular triangulation* is a map with one inner *simple* face of degree p and the other faces of degree 3.

Separation. A p-annular triangulation is *separated* if there is a cycle of length 3 separating the outer face and the p-gonal face.
Any p-annular d-angulation A decomposes into a 3-annular triangulation (i.e., triangulation with a marked inner triangle) and a non-separated p-annular triangulation.



Orientation. Let $p > 3$. Any p-annular triangulation A admits a unique minimal orientation such that the p-gon is a directed cycle and any vertex not on the p-gon has indegree 3. This orientation is in $\mathcal{O}$ if and only if A is non-separated.

Theorem. The master bijection Φ_+ induces a bijection between simple *non-separated* p-annular triangulations and mobiles such that
- black vertices have degree 3 except a special vertex of degree p with no buds.
- white vertices have degree 3 except neighbors of the special vertex having degrees summing to $2p-3$.



Counting results (recovering Brown [4,5]). Let $t_{p,n}$ be the number of simple p-gonal triangulations with $n+p$ vertices rooted in the p-gonal face. The formal series $T_p(x) = \sum_{n\geq 0}(2n+p-2)\,t_{p,n}x^n$ satisfies $T_p(x) := \binom{2p-4}{p-3}\,u^{2p-3}$, where $u = 1 + x\,u^4$.
Consequently, the Lagrange inversion formula gives: $t_{p,n} = \frac{2(2p-3)!}{(p-1)!(p-3)!}\frac{(4n+2p-5)!}{n!(3n+2p-3)!}$.

The same strategy applies to quadrangulations (and pentagulations, etc. [*]).

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