Harmonics for Twisted Steenrod Operators.

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S_{n}-Harmonic Polynomials

Solutions of the system

\[(\partial_1 + \ldots + \partial_n) f(x_1, \ldots, x_n) = 0\]
\[(\partial_1^2 + \ldots + \partial_n^2) f(x_1, \ldots, x_n) = 0\]
\[\vdots\]
\[(\partial_1^n + \ldots + \partial_n^n) f(x_1, \ldots, x_n) = 0\]

are said to be \(S_{n}\)-harmonic polynomials. Here \(\partial_i := \frac{\partial}{\partial x_i}\).

They can be characterized as all the linear combinations of

\[\partial_1^{k_1} \partial_2^{k_2} \ldots \partial_n^{k_n} \prod_{i<j} (x_i - x_j)\]
Graded $\mathfrak{S}_n$-Module

The space $H_{\mathfrak{S}_n}$ of harmonic polynomials for the symmetric group is a graded $\mathfrak{S}_n$-module, i.e.:

$$H_{\mathfrak{S}_n} \simeq \bigoplus_{d \geq 0} \pi_d(H_{\mathfrak{S}_n}),$$

where $\pi_d$ is the linear projection sending polynomials to their degree $d$ homogeneous component.

Recall that the group $\mathfrak{S}_n$ acts on polynomials in $n$ variables

$$x = x_1, x_2, \ldots, x_n$$

by permuting variables:

$$\sigma \cdot x_i = x_{\sigma(i)}.$$
Graded Irreducible Decomposition

From the now classical decomposition

\[ \mathbb{Q}[x] \cong \mathbb{Q}[x]^{
abla_n} \otimes \mathbb{H}_{\nabla_n}, \]

as graded \( \nabla_n \)-modules, we get

\[ \mathbb{Q}[x] \cong \bigoplus_{\ell(\mu) \leq n} \bigoplus_{\lambda \vdash n \text{ sh}(\tau) = \lambda} e_\mu \mathcal{V}_\tau, \]

where:

(1) \( \mathcal{V}_\tau \) is some copy of an irreducible representation of \( \nabla_n \), of Frobenius characteristic \( s_\lambda \), in the homogeneous component, in \( \mathbb{H}_{\nabla_n} \), of degree equal to the cocharge, \( \text{co}(\tau) \), of \( \tau \).

(2) The indices \( \tau \) run over the set of standard tableaux of shape \( \lambda \).

(3) Finally, \( e_\mu \) denotes the elementary polynomials in the variables \( x \), considered as linear operator on \( \mathbb{Q}[x] \).
Hilbert Series

Recall that we have

$$\sum_{d \geq 0} \dim(\pi_d(\mathbb{Q}[x])) t^d = \left(\frac{1}{1-t}\right)^n$$

and the previous decomposition gives

$$\left(\frac{1}{1-t}\right)^n = \prod_{i=1}^{n} \frac{1}{1-t^i} \sum_{\lambda \vdash n} \sum_{\text{sh}((\tau)) = \lambda} n_{\lambda} t^{\text{co}(\tau)}$$

with $n_{\lambda}$ equal to the number of standard tableaux of shape $\lambda$. Recall that

$$n(\lambda) := \sum_{i} (i-1) \lambda_i.$$

is the smallest possible value for the cocharge of a standard tableaux of shape $\lambda$. 
Twisted Steenrod Operators and Associated Harmonics

\[ D_{k;q} := \sum_{i=1}^{n} q x_i \partial_{i}^{k+1} + \partial_{i}^{k} \quad \text{H}_{n;q} := \{ f | D_{k;q} f = 0, \ \forall k \geq 1 \} \]

\[ \tilde{D}_{k} := \sum_{i=1}^{n} x_i \partial_{i}^{k+1} \quad \tilde{\text{H}}_{x} := \{ f | \tilde{D}_{k} f = 0, \ \forall k \geq 1 \} \]

\[ \hat{D}_{k} := \sum_{i=1}^{n} x_i \partial_{i}^{k+1} + (k + 1) \partial_{i}^{k} \quad \hat{\text{H}}_{x} := \{ f | \hat{D}_{k} f = 0, \ \forall k \geq 1 \} \]

\[ D_{k} := \sum_{i=1}^{n} a_i \partial_{i}^{k} \quad \text{H}_{n} := \{ f | D_{k} f = 0, \ \forall k \geq 1 \} \]

All of these spaces are homogeneous, and we write

\[ \mathcal{H}_{n;q}(t), \quad \tilde{\mathcal{H}}_{n}(t), \quad \hat{\mathcal{H}}_{n}(t), \quad \text{and} \quad \mathcal{H}_{n}(t), \]

for the respective Hilbert series.
Conjecture (HT). The space $H_{n;q}$ is isomorphic, as a graded $S_n$-module, to the space of $S_n$-harmonic polynomials, for “generic” values of $q$.

In fact

$$H_{n;q} = \{ f \mid D_{1;q} f = 0, \text{ and } D_{2;q} f = 0 \}$$

since

$$[D_{k;q}, D_{j;q}] = q(k - j)D_{k+j;q}$$
General Conjecture

For the general operators

\[ \overline{D}_1 := \sum_{i=1}^{n} b_i x_i \partial_i^2 + a_i \partial_i, \quad \text{and} \]

\[ \overline{D}_2 := \sum_{i=1}^{n} d_i x_i \partial_i^3 + c_i \partial_i^2, \]

set

\[ \overline{H}_n = \{ f \mid \overline{D}_1 f = 0, \quad \text{and} \quad \overline{D}_2 f = 0 \}. \]

Then

**Conjecture (B).** There is a graded space isomorphism between the space \( \overline{H}_n \) and the space of \( S_n \)-harmonic polynomials, for “generic” values of \( a_i, b_i, c_i, \) and \( d_i \).
Dual Point of View

For the scalar product on $\mathbb{Q}[x]$ defined by

$$\langle x^a, x^b \rangle := \begin{cases} a! & \text{if } x^a = x^b, \\ 0 & \text{otherwise,} \end{cases}$$

for two monomials $x^a$ and $x^b$ (in vector notation), with $a!$ standing for $a_1!a_2!\cdots a_n!$, we easily check that

$$\langle x^i_k \partial^j_i f, g \rangle = \langle f, x^j_i \partial^k_i g \rangle$$

thus we get the dual operators

$$D^*_{k; q} = \sum_{i=1}^{n} q x^{k+1}_i \partial_i + x^k_i$$
Hit Polynomials

Following R. Wood we say that a polynomial is hit, for the operators $D^*_{k; q}$, if it can be expressed in the form

$$f(x) = \sum_{k \geq 1} D^*_{k; q} g_k(x),$$

for some polynomials $g_k$. We write $\mathcal{C}_{n; q}$ for the graded quotient of the space of polynomial by the subspace of hit-polynomials for the operators $D^*_{k; q}$. Likewise, we write

$$\tilde{\mathcal{C}}_n, \quad \text{and} \quad \hat{\mathcal{C}}_n$$

for the spaces respectively associated to the operators

$$\tilde{D}^*_k := \sum_{i=1}^n x_i^{k+1} \partial_i, \quad \text{and} \quad \hat{D}^*_k := \sum_{i=1}^n x_i^{k+1} \partial_i + (k + 1)x_i^k.$$
Wood’s Conjecture

**Conjecture (W).** The space $\widehat{C}_n$ contains a copy of the regular representation spanned by the monomials

$$e_x x^a, \quad \text{with} \quad a = (a_1, \ldots, a_n), \quad 0 \leq a_i < i,$$

with $e_x = x_1 x_2 \cdots x_n$.

In fact, we will see that the entire space can be described as follows.

**Conjecture (BGW).** The space $\widehat{C}_n$ affords the basis

$$e_y y^a, \quad \text{with} \quad a = (a_1, \ldots, a_k), \quad 0 \leq a_i < i,$$

with $e_y = y_1 y_2 \cdots y_k$, $k$ varying from 0 to $n$, and $y$ varying in all $k$-subsets of $x$.

Clearly the spaces $\mathcal{C}_{n; q}, \widehat{C}_n, \hat{C}_n$ are respectively isomorphic, as graded $\mathfrak{S}_n$-modules, to the spaces $H_{n; q}, \hat{H}_n, \hat{H}_n$. 
Example

For the space \( \mathcal{C}_3 \) we have the basis

\[
1, \\
x_1, x_2, x_3, \\
x_1x_2, x_1x_2^2, \\
x_1x_3, x_1x_3^2, \\
x_2x_3, x_2x_3^2, \\
x_1x_2x_3, x_1x_2^2x_3, x_1x_2x_3^2, x_1x_2^2x_3^2, x_1x_2x_3^3, x_1x_2^2x_3^3.
\]

Modulo the conjecture, the associated Hilbert series is

\[
\tilde{H}_n(t) = \sum_{k=0}^{n} \binom{n}{k} t^k [k]_t!.
\]
Graded Frobenius Characteristic

Recall that the *graded Frobenius characteristic* of an invariant homogeneous subspace

\[ \mathcal{V} = \bigoplus_{d \geq 0} \mathcal{V}_d, \]

of \( \mathbb{Q}[x] \) is

\[ \mathcal{F}_\mathcal{V}(t) := \sum_{d \geq 0} t^d \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \chi^{\mathcal{V}_d}(\sigma)p_{\lambda(\sigma)} \]

Since the associated operators are symmetric, the spaces \( \mathcal{H}_{n;q}, \tilde{\mathcal{H}}_n, \hat{\mathcal{H}}_n \) are invariant homogeneous spaces, we have corresponding

\[ \mathcal{F}_{n;q}(t), \quad \tilde{\mathcal{F}}_n(t), \quad \text{and} \quad \hat{\mathcal{F}}_n(t), \]

graded Frobenius characteristics.
First Results

We have the following

**Theorem (1).** If \( q \) is considered as a formal parameter, then the space \( \mathcal{H}_{x;q} \) is isomorphic, as a graded \( \mathfrak{S}_n \)-module, to a submodule of the \( \mathfrak{S}_n \)-harmonics.

**Theorem (2).** Let the Hilbert series of \( \mathcal{H}_{x;q} \) be

\[
\mathcal{H}_{x;q}(t) = \sum_{d \geq 0} c_{d,n} t^d,
\]

then

\[
c_{d,n} = [n]_t! \bigg|_{t^d}, \quad \forall d \leq n.
\]

**Theorem (3).** The space of tilde-harmonics has the direct sum decomposition

\[
\tilde{\mathcal{H}}_x = \bigoplus_{y \subseteq x} e_y \hat{\mathcal{H}}_y.
\]
Proof of Theorem 3

Decompose $f$ in $\mathbb{Q}[x]$ in the form

$$f = \sum_{y \subseteq x} e_y f_y$$

with $f_y$ in $\mathbb{Q}[y]$. Then one checks that $f$ is in $\tilde{\mathcal{H}}_x$ if and only if all $f_y$ are chosen to lie in $\hat{\mathcal{H}}_y$, using the operator identity

$$\tilde{D}_k e_x = e_x \hat{D}_k.$$

In other words, we get

$$\bigoplus_{y \subseteq x} e_y \hat{\mathcal{H}}_y = \tilde{\mathcal{H}}_x,$$

thus finishing the proof.
Implication for the Frobenius

It follows from this proof that the graded Frobenius characteristic of $\tilde{H}_x^a$ is given by the symmetric function

$$\tilde{F}_a(t) = \sum_{k=0}^{n} t^k \hat{F}_a(t) h_{n-k}.$$ 

Here $a$ stands for the characteristic function for selection of some subset of indices for which we set

$$\tilde{H}_x^a := \{ f \mid \tilde{D}_k f = 0, \text{ if } a(k) = 0 \},$$

$$\hat{H}_x^a := \{ f \mid \hat{D}_k f = 0, \text{ if } a(k) = 0 \}.$$
Kernel of $D_k$

The Hilbert series of the kernel of

$$D_k := \sum_{i=1}^{n} b_i x_i \partial_i^{k+1} + a_i \partial_i^k$$

is

$$(1 + t + \ldots + t^{k-1}) \left( \frac{1}{1 - t} \right)^{n-1},$$

and we have an explicit description of it. For $k = 1$, the elements of the kernel take the form

$$f = \sum_{r} c_r (y^r + \Psi_1(y^r)),$$

where, setting $x = x_n$, $a = a_n$, $b = b_n$ and $y = x_1, \ldots, x_{n-1}$; we have

$$\Psi_1(g) := \sum_{m \geq 1} (-1)^m \frac{D_1^m(g)}{[a; b]_m} \frac{x^m}{m!}.$$
One Generic Case

We have the following

**Theorem (4).** For all choices of $a_i$'s such that

$$\sum_{k \in K} a_k \neq 0, \quad \forall K \subseteq \{1, \ldots, n\}, \ K \neq \emptyset,$$

the Hilbert series of the space

$$\{ f \mid \sum_{i=1}^{n} a_i \partial_i^{m+j} f = 0, \ \text{for} \ 1 \leq j \leq n \},$$

is

$$[m+n \choose n] [n]! t^n$$

for all $m \geq 0$. 
Proof of Theorem 4

To prove the theorem, we use the fact that

**Proposition.** Polynomials \( \theta_1(x), \theta_2(x), \ldots, \theta_n(x) \) form a regular sequence in \( \mathbb{Q}[x] \) if and only if the system of equations

\[
\theta_1(x) = 0, \quad \theta_2(x) = 0, \quad \ldots, \quad \theta_n(x) = 0
\]

has, for \( x \in \mathbb{Q}^n \), the unique solution

\[
x_1 = 0, \quad x_2 = 0, \quad \ldots, \quad x_n = 0.
\]

For our case, we formulate this in the format

\[
\begin{pmatrix}
x_1^{m+1} & x_2^{m+1} & \cdots & x_n^{m+1} \\
x_1^{m+2} & x_2^{m+2} & \cdots & x_n^{m+2} \\
\vdots & \vdots & \ddots & \vdots \\
x_1^{m+n} & x_2^{m+n} & \cdots & x_n^{m+n}
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2 \\
\vdots \\
a_n
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
\vdots \\
0
\end{pmatrix}
\]
An Hyperplane Arrangement

For the hyperplane arrangement

\[ \prod_{\substack{K \subseteq \{1, \ldots, n\} \atop K \neq \emptyset}} \left( \sum_{k \in K} a_k \right) = 0, \]

the number of chambers are

1, 2, 6, 32, 370, 11292, \ldots

For \( n = 3 \) we get
Diagonal version

Conjecture (B). The space corresponding to the set of common zeros of the operators

$$\sum_{i=1}^{n} a_i \partial^k x_i \partial^j y_i,$$

for all $k, j \in \mathbb{N}$ such that $k + j > 0$, is of dimension $(n + 1)^{n-1}$, whenever we have

$$\sum_{k \in K} a_k \neq 0,$$

for all nonempty subsets $K$ of $\{1, \ldots, n\}$.

A stronger statement can be made in term of bigraded Hilbert series, and several sets of variables.
References/see arXiv:0812.3566

