Harmonics for Twisted Steenrod Operators.

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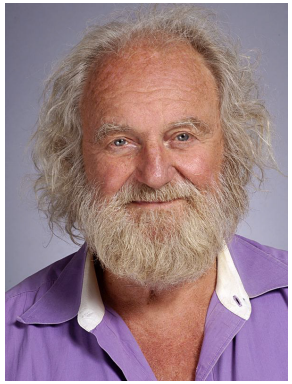
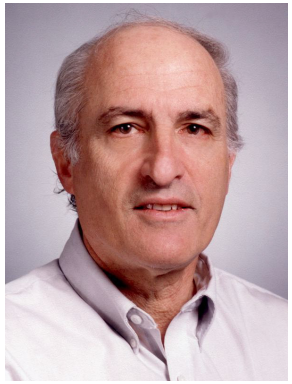




$\mathfrak{S}_n$-Harmonic Polynomials

Solutions of the system

$$\begin{aligned}
(\partial_1 + \ldots + \partial_n) f(x_1, \ldots, x_n) &= 0 \\
(\partial_1^2 + \ldots + \partial_n^2) f(x_1, \ldots, x_n) &= 0 \\
&\vdots \\
(\partial_1^n + \ldots + \partial_n^n) f(x_1, \ldots, x_n) &= 0
\end{aligned}$$

are said to be *$\mathfrak{S}_n$-harmonic polynomials*. Here $\partial_i := \frac{\partial}{\partial x_i}$.

They can be characterized as all the linear combinations of

$$\partial_1^{k_1} \partial_2^{k_2} \cdots \partial_n^{k_2} \prod_{i<j} (x_i - x_j)$$

Graded $\mathfrak{S}_n$-Module

The space $\mathbf{H}_{\mathfrak{S}_n}$ of harmonic polynomials for the symmetric group is a *graded $\mathfrak{S}_n$-module*, i.e.:

$$\mathbf{H}_{\mathfrak{S}_n} \simeq \bigoplus_{d \geq 0} \pi_d(\mathbf{H}_{\mathfrak{S}_n}),$$

where π_d is the linear projection sending polynomials to there degree d homogeneous component.

Recall that the group $\mathfrak{S}_n$ acts on polynomials in n variables

$$\mathbf{x} = x_1, x_2, \ldots, x_n$$

by permuting variables:

$$\sigma \cdot x_i = x_{\sigma(i)}.$$

Graded Irreducible Decomposition

From the now classical decomposition

$$\mathbb{Q}[\mathbf{x}] \simeq \mathbb{Q}[\mathbf{x}]^{\mathfrak{S}_n} \otimes \mathbf{H}_{\mathfrak{S}_n},$$

as graded $\mathfrak{S}_n$-modules, we get

$$\mathbb{Q}[\mathbf{x}] \simeq \bigoplus_{\ell(\mu)\leq n} \bigoplus_{\lambda\vdash n} \bigoplus_{\mathrm{sh}(\tau)=\lambda} e_\mu \mathcal{V}_\tau,$$

where:

(1) $\mathcal{V}_\tau$ is some copy of an irreducible representation of $\mathfrak{S}_n$, of Frobenius characteristic s_λ, in the homogeneous component, in $\mathbf{H}_{\mathfrak{S}_n}$, of degree equal to the *cocharge*, $\mathrm{co}(\tau)$, of τ.

(2) The indices τ run over the set of standard tableaux of shape λ.

(3) Finally, e_μ denotes the elementary polynomials in the variables $\mathbf{x}$, considered as linear operator on $\mathbb{Q}[\mathbf{x}]$.

Hilbert Series

Recall that we have

$$\sum_{d\geq 0} \dim(\pi_d(\mathbb{Q}[\mathbf{x}]))t^d = \left(\frac{1}{1-t}\right)^n$$

and the previous decomposition gives

$$\left(\frac{1}{1-t}\right)^n = \prod_{i=1}^{n} \frac{1}{1-t^i} \sum_{\lambda\vdash n} \sum_{\mathrm{sh}(\tau)=\lambda} n_\lambda t^{\mathrm{co}(\tau)}$$

with n_λ equal to the number of standard tableaux of shape λ. Recall that

$$n(\lambda) := \sum_i (i-1)\,\lambda_i.$$

is the smallest possible value for the cocharge of a standard tableaux of shape λ.

Twisted Steenrod Operators and Associated Harmonics

$$D_{k;q} := \sum_{i=1}^{n} q\, x_i \partial_i^{k+1} + \partial_i^k \qquad \mathbf{H}_{n;q} := \{f \mid D_{k;q} f = 0,\ \forall k \geq 1\}$$

$$\widetilde{D}_k := \sum_{i=1}^{n} x_i \partial_i^{k+1} \qquad \widetilde{\mathbf{H}}_{\mathbf{x}} := \{f \mid \widetilde{D}_k f = 0,\ \forall k \geq 1\}$$

$$\widehat{D}_k := \sum_{i=1}^{n} x_i \partial_i^{k+1} + (k+1)\partial_i^k \qquad \widehat{\mathbf{H}}_{\mathbf{x}} := \{f \mid \widehat{D}_k f = 0,\ \forall k \geq 1\}$$

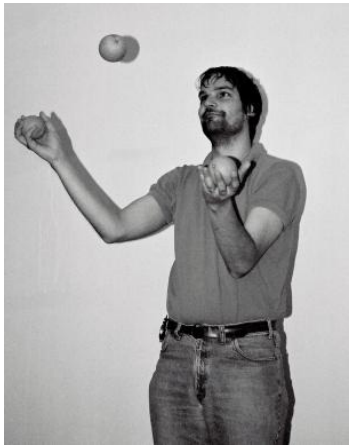
$$D_k := \sum_{i=1}^{n} a_i\, \partial_i^k \qquad \mathbf{H}_n := \{f \mid D_k f = 0,\ \forall k \geq 1\}$$

All of these spaces are homogeneous, and we write

$$\mathcal{H}_{n;q}(t), \qquad \widetilde{\mathcal{H}}_n(t), \qquad \widehat{\mathcal{H}}_n(t), \qquad \text{and} \qquad \mathcal{H}_n(t),$$

for the respective Hilbert series.

Conjecture of Hivert-Thiéry



Conjecture (HT). *The space* $\mathbf{H}_{n;q}$ *is isomorphic, as a graded* $\mathfrak{S}_n$*-module, to the space of* $\mathfrak{S}_n$*-harmonic polynomials, for "generic" values of* q.

In fact

$$\mathbf{H}_{n;q} = \{f \mid D_{1;q}\, f = 0, \quad \text{and} \quad D_{2;q}\, f = 0\}$$

since

$$[D_{k;q}, D_{j;q}] = q(k-j)D_{k+j;q}$$

General Conjecture

For the general operators

$$\overline{D}_1 \quad := \quad \sum_{i=1}^{n} b_i\, x_i \partial_i^2 + a_i\, \partial_i, \quad \text{and}$$

$$\overline{D}_2 \quad := \quad \sum_{i=1}^{n} d_i\, x_i \partial_i^3 + c_i\, \partial_i^2,$$

set

$$\overline{\mathbf{H}}_n = \{ f \mid \overline{D}_1\, f = 0, \quad \text{and} \quad \overline{D}_2\, f = 0 \}.$$

Then

Conjecture (B). *There is a graded space isomorphism between the space* $\overline{\mathbf{H}}_n$ *and the space of* $\mathfrak{S}_n$*-harmonic polynomials, for "generic" values of* a_i, b_i, c_i, *and* d_i.

Dual Point of View

For the *scalar product* on $\mathbb{Q}[\mathbf{x}]$ defined by

$$\langle \mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}} \rangle := \begin{cases} \mathbf{a}! & \text{if, } \mathbf{x}^{\mathbf{a}} = \mathbf{x}^{\mathbf{b}}, \\ 0 & \text{otherwise,} \end{cases}$$

for two monomials $\mathbf{x}^{\mathbf{a}}$ and $\mathbf{x}^{\mathbf{b}}$ (in vector notation), with $\mathbf{a}!$ standing for $a_1! a_2! \cdots a_n!$, we easily check that

$$\langle x_i^k \partial_i^j f, g \rangle = \langle f, x_i^j \partial_i^k g \rangle$$

thus we get the dual operators

$$D^*_{k;q} = \sum_{i=1}^{n} q\, x_i^{k+1} \partial_i + x_i^k$$

Hit Polynomials

Following R. Wood we say that a polynomial is *hit*, for the operators $D^*_{k;q}$, if it can be expressed in the form

$$f(\mathbf{x}) = \sum_{k\geq 1} D^*_{k;q}\, g_k(\mathbf{x}),$$

for some polynomials g_k. We write $\mathcal{C}_{n;q}$ for the *graded quotient of the space of polynomial by the subspace of hit-polynomials* for the operators $D^*_{k;q}$. Likewise, we write

$$\widetilde{\mathcal{C}}_n, \qquad \text{and} \qquad \widehat{\mathcal{C}}_n$$

for the spaces respectively associated to the operators

$$\widetilde{D}^*_k := \sum_{i=1}^{n} x_i^{k+1}\partial_i, \qquad \text{and} \qquad \widehat{D}^*_k := \sum_{i=1}^{n} x_i^{k+1}\partial_i + (k+1)x_i^k.$$

Wood's Conjecture

Conjecture (W). *The space* $\widetilde{\mathcal{C}}_n$ *contains a copy of the regular representation spanned by the monomials*

$$e_{\mathbf{x}}\,\mathbf{x}^{\mathbf{a}}, \qquad \text{with} \qquad \mathbf{a} = (a_1, \ldots, a_n), \qquad 0 \le a_i < i,$$

with $e_{\mathbf{x}} = x_1 x_2 \cdots x_n$.

In fact, we will see that the entire space can be described as follows.

Conjecture (BGW). *The space* $\widetilde{\mathcal{C}}_n$ *affords the basis*

$$e_{\mathbf{y}}\,\mathbf{y}^{\mathbf{a}}, \qquad \text{with} \qquad \mathbf{a} = (a_1, \ldots, a_k), \qquad 0 \le a_i < i,$$

with $e_{\mathbf{y}} = y_1 y_2 \cdots y_k$, k *varying from* 0 *to* n, *and* $\mathbf{y}$ *varying in all* k*-subsets of* $\mathbf{x}$.

Clearly the spaces $\mathcal{C}_{n;q}$, $\widetilde{\mathcal{C}}_n$, $\widehat{\mathcal{C}}_n$ are respectively isomorphic, as graded $\mathfrak{S}_n$-modules, to the spaces $\mathbf{H}_{n;q}$, $\widetilde{\mathbf{H}}_n$, $\widehat{\mathbf{H}}_n$.

Example

For the space $\widetilde{\mathcal{C}}_3$ we have the basis

$$
\begin{aligned}
&1,\\
&x_1, x_2, x_3,\\
&x_1x_2, x_1x_2^2,\\
&x_1x_3, x_1x_3^2,\\
&x_2x_3, x_2x_3^2,\\
&x_1x_2x_3, x_1x_2^2x_3, x_1x_2x_3^2, x_1x_2^2x_3^2, x_1x_2x_3^3, x_1x_2^2x_3^3.
\end{aligned}
$$

Modulo the conjecture, the associated Hilbert series is

$$\widetilde{\mathcal{H}}_n(t) = \sum_{k=0}^{n} \binom{n}{k} t^k [k]_t!.$$

Graded Frobenius Characteristic

Recall that the *graded Frobenius characteristic* of an invariant homogeneous subspace

$$\mathcal{V} = \bigoplus_{d \geq 0} \mathcal{V}_d,$$

of $\mathbb{Q}[\mathbf{x}]$ is

$$\mathcal{F}_{\mathcal{V}}(t) := \sum_{d \geq 0} t^d \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \chi^{\mathcal{V}_d}(\sigma) p_{\lambda(\sigma)}$$

Since the associated operators are symmetric, the spaces $\mathbf{H}_{n;q}$, $\widetilde{\mathbf{H}}_n$, $\widehat{\mathbf{H}}_n$ are invariant homogeneous spaces, we have corresponding

$$\mathcal{F}_{n;q}(t), \qquad \widetilde{\mathcal{F}}_n(t), \qquad \text{and} \qquad \widehat{\mathcal{F}}_n(t),$$

graded Frobenius characteristics.

First Results

We have the following

Theorem (1). *If q is considered as a formal parameter, then the space $\mathbf{H}_{\mathbf{x};q}$ is isomorphic, as a graded $\mathfrak{S}_n$-module, to a submodule of the $\mathfrak{S}_n$-harmonics.*

Theorem (2). *Let the Hilbert series of $\mathbf{H}_{\mathbf{x};q}$ be*

$$\mathcal{H}_{\mathbf{x};q}(t) = \sum_{d \geq 0} c_{d,n}\, t^d,$$

then

$$c_{d,n} = [n]_t!\Big|_{t^d}, \qquad \forall d \leq n.$$

Theorem (3). *The space of tilde-harmonics has the direct sum decomposition*

$$\widetilde{\mathbf{H}}_{\mathbf{x}} = \bigoplus_{\mathbf{y} \subseteq \mathbf{x}} e_{\mathbf{y}} \widehat{\mathbf{H}}_{\mathbf{y}}.$$

Proof of Theorem 3

Decompose f in $\mathbb{Q}[\mathbf{x}]$ in the form

$$f = \sum_{\mathbf{y} \subseteq \mathbf{x}} e_{\mathbf{y}} f_{\mathbf{y}}$$

with $f_{\mathbf{y}}$ in $\mathbb{Q}[\mathbf{y}]$. Then one checks that f is in $\widetilde{\mathbf{H}}_{\mathbf{x}}$ if and only if all $f_{\mathbf{y}}$ are chosen to lie in $\widehat{\mathbf{H}}_{\mathbf{y}}$, using the operator identity

$$\widetilde{D}_k \, e_{\mathbf{x}} = e_{\mathbf{x}} \, \widehat{D}_k .$$

In other words, we get

$$\bigoplus_{\mathbf{y} \subseteq \mathbf{x}} e_{\mathbf{y}} \widehat{\mathbf{H}}_{\mathbf{y}} \;=\; \widetilde{\mathbf{H}}_{\mathbf{x}},$$

thus finishing the proof.

Implication for the Frobenius

It follows from this proof that the graded Frobenius characteristic of $\widetilde{\mathbf{H}}^{\mathbf{a}}_{\mathbf{x}}$ is given by the symmetric function

$$\widetilde{\mathcal{F}}_{\mathbf{a}}(t) = \sum_{k=0}^{n} t^k \, \widehat{\mathcal{F}}_{\mathbf{a}}(t) \, h_{n-k}.$$

Here $\mathbf{a}$ stands for the characteristic function for selection of some subset of indices for which we set

$$\widetilde{\mathbf{H}}^{\mathbf{a}}_{\mathbf{x}} \quad := \quad \{f \mid \widetilde{D}_k f = 0, \text{ if } \mathbf{a}(k) = 0\}, \qquad \text{and}$$

$$\widehat{\mathbf{H}}^{\mathbf{a}}_{\mathbf{x}} \quad := \quad \{f \mid \widehat{D}_k f = 0, \text{ if } \mathbf{a}(k) = 0\}.$$

Kernel of D_k

The Hilbert series of the kernel of

$$D_k := \sum_{i=1}^{n} b_i \, x_i \partial_i^{k+1} + a_i \, \partial_I^k$$

is

$$(1 + t + \ldots + t^{k-1}) \left(\frac{1}{1-t} \right)^{n-1},$$

and we have an explicit description of it. For $k = 1$, the elements of the kernel take the form

$$f = \sum_{\mathbf{r}} c_{\mathbf{r}} (\mathbf{y}^{\mathbf{r}} + \Psi_1(\mathbf{y}^{\mathbf{r}})),$$

where, setting $x = x_n$, $a{=}a_n$, $b{=}b_n$ and $\mathbf{y} = x_1, \ldots, x_{n-1}$; we have

$$\Psi_1(g) := \sum_{m \geq 1} (-1)^m \frac{D_1^m(g)}{[a;b]_m} \frac{x^m}{m!}.$$

One Generic Case

We have the following

Theorem (4). *For all choices of a_i's such that*

$$\sum_{k \in K} a_k \neq 0, \quad \forall K \subseteq \{1, \ldots, n\},\ K \neq \emptyset,$$

the Hilbert series of the space

$$\{f \mid \sum_{i=1}^{n} a_i\ \partial_i^{m+j} f = 0, \quad \text{for} \quad 1 \leq j \leq n\},$$

is

$$\begin{bmatrix} m+n \\ n \end{bmatrix}_t [n]!_t$$

for all $m \geq 0$.

Proof of Theorem 4

To prove the theorem, we use the fact that

Proposition. *Polynomials* $\theta_1(\mathbf{x}), \theta_2(\mathbf{x}), \ldots, \theta_n(\mathbf{x})$ *form a regular sequence in* $\mathbb{Q}[\mathbf{x}]$ *if and only if the system of equations*

$$\theta_1(\mathbf{x}) = 0\,,\ \theta_2(\mathbf{x}) = 0\,,\ \ldots\,,\ \theta_n(\mathbf{x}) = 0$$

has, for $\mathbf{x} \in \mathbb{Q}^n$*, the unique solution*

$$x_1 = 0\,,\ x_2 = 0\,,\ \ldots\,,\ x_n = 0.$$

For our case, we formulate this in the format

$$\begin{pmatrix} x_1^{m+1} & x_2^{m+1} & \ldots & x_n^{m+1} \\ x_1^{m+2} & x_2^{m+2} & \ldots & x_n^{m+2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{m+n} & x_2^{m+n} & \ldots & x_n^{m+n} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

An Hyperplane Arrangement

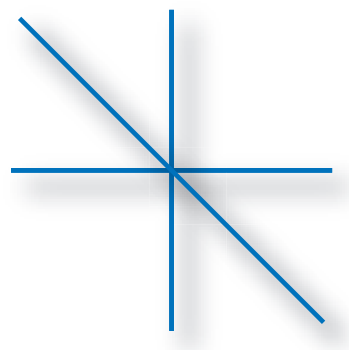
For the hyperplane arrangement

$$\prod_{\substack{K \subseteq \{1,\ldots,n\} \\ K \neq \emptyset}} \Big(\sum_{k \in K} a_k \Big) = 0,$$

the number of chambers are

$$1, 2, 6, 32, 370, 11292, \ldots$$

For $n = 3$ we get



Diagonal version

Conjecture (B). *The space corresponding to the set of common zeros of the operators*

$$\sum_{i=1}^{n} a_i \, \partial_{x_i}^k \partial_{y_i}^j,$$

for all $k, j \in \mathbb{N}$ *such that* $k + j > 0$, *is of dimension* $(n+1)^{n-1}$, *whenever we have*

$$\sum_{k \in K} a_k \neq 0,$$

for all nonempty subsets K *of* $\{1, \ldots, n\}$.

A stronger statement can be made in term of bigraded Hilbert series, and several sets of variables.

References/see arXiv:0812.3566

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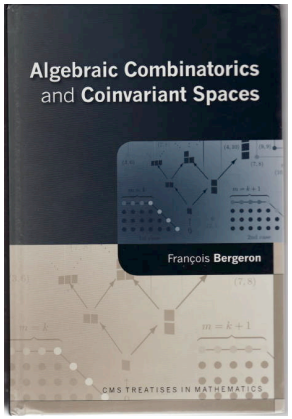

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