Weakly Directed Walks

A New Class of Self-Avoiding Walks

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Introduction & Definitions

Self-avoiding walks (SAW)

Definition 1

A lattice walk is *self-avoiding* if it does not visit twice the same vertex.

Conjecture 2

The number c_n of SAW of length n and their average end-to-end distance D_n verify

$$c_n \sim \alpha \mu^n n^{11/32} \quad \textit{and} \quad D_n \sim \kappa n^{3/4} \quad \textit{with } \mu \approx 2.64.$$

Motivation: Find a subclass of SAW both large and conceptually pleasant. *Previous record: prudent walks*, with $\mu \approx 2.48$.

Weakly directed bridges

Notation: A NES-walk (for instance) is a walk that takes only N, E, and S steps.

Definition 3

A walk $v_0 \cdots v_n$ is a *bridge* if every vertex $v \neq v_n$ satisfies $h(v_0) \leq h(v) < h(v_n)$. A nonempty bridge is *irreducible* if it does not factor into two nonempty bridges.

A self-avoiding bridge is *weakly directed* if every irreducible bridge is either a NES-walk or a NWS-walk (Figure 1).

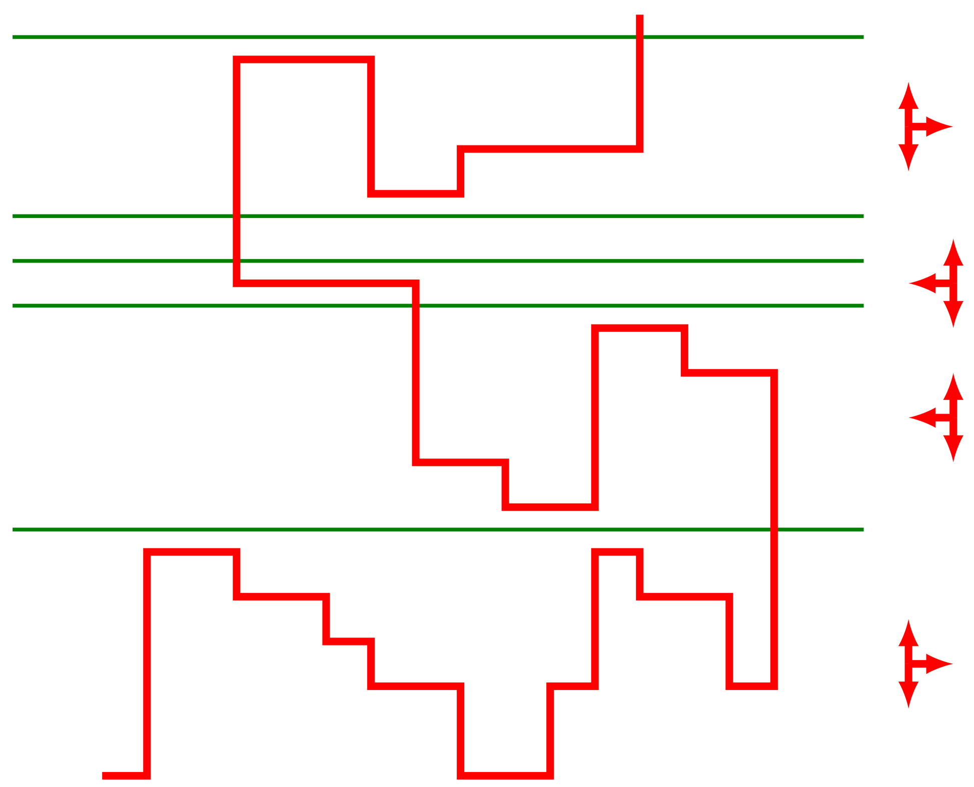

Figure 1: A weakly directed bridge factored into five irreducible bridges.

Enumeration

Link with partially directed bridges

Notation: Let:

- $B(t)$ be the generating function of NES-bridges;
- $I(t)$ be the generating function of NES-irreducible bridges;
- $W(t)$ be the generating function of weakly directed bridges.

The decomposition into irreducible bridges yields:

$$B(t) = \frac{1}{1 - I(t)}; \qquad W(t) = \frac{1}{1 - \big(2I(t) - t\big)}.$$

Therefore:

$$W(t) = \frac{1}{1 - \frac{2B(t)}{1+B(t)} + t}.$$

Enumeration of partially directed bridges

Theorem 4

Let $k \geq 0$. The generating function of NES-bridges of height k is

$$B_k(t) = \frac{t^k}{G_k(t)},$$

where $G_k(t)$ is the sequence of polynomials defined by

$$G_0 = 1; \qquad G_1 = 1 - t;$$
$$G_k = (1 - t + t^2 + t^3)G_{k-1} - t^2 G_{k-2} \qquad \textit{for } k \geq 2.$$

Two different proofs of this result are outlined below (details on demand!).

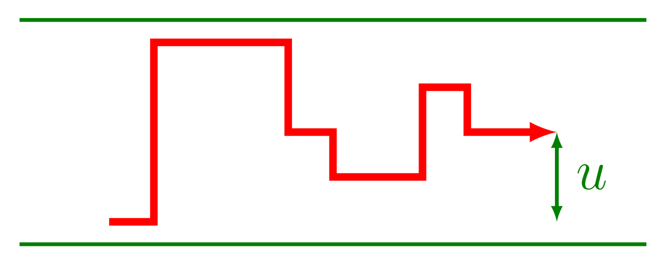

Figure 2: First proof of Theorem 4. Let $T_k(t, u)$ be the generating function of walks in a strip of height $k - 1$, with u accounting for the final height. The value of T_k is derived using the *kernel method*, and B_k follows. [1]

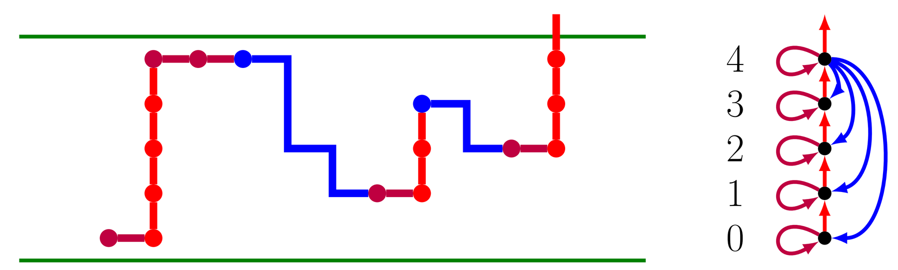

Figure 3: Second proof of Theorem 4. A NES-bridge of height k is seen as an arbitrary sequence of *generalized steps*, and thus as a walk on a graph with vertices $\{0, \ldots, k-1\}$. Such walks are enumerated using *heaps of cycles* techniques. [3]

Asymptotics & Random Sampling

Nature of the series & asymptotics

Proposition 5

The generating functions $B(t)$ and $W(t)$ both have a complex singularity structure and are not D-finite.

The number w_n of weakly directed bridges and their average end-to-end distance E_n are asymptotically

$$w_n \sim \alpha \mu^n \quad \textit{and} \quad E_n \sim \kappa n, \quad \textit{with } \mu \approx 2.5447.$$

The growth constant is larger than for prudent walks, but the average end-to-end distance is still linear.

Random sampling

Proposition 6

It is possible to sample a weakly directed walk with approximate length n in time $O(n)$, using a Boltzmann sampler. [2]

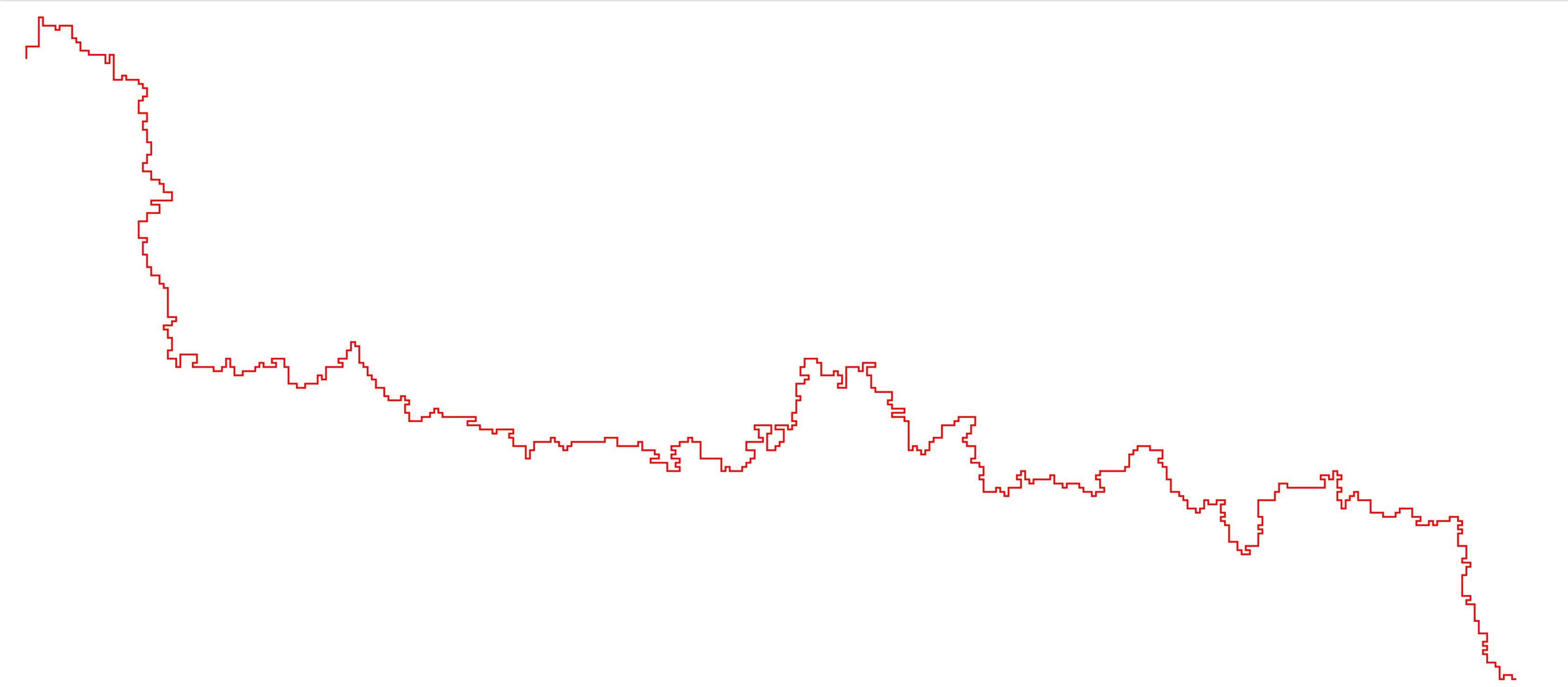

Figure 4: A random weakly directed bridge of length 1001, turned by 90°.

References

[1] M. Bousquet-Mélou and Y. Ponty. Culminating paths. *Discrete Math. Theoret. Comput. Sci., 10(2), 2008.* arXiv:0706.0694.

[2] Ph. Duchon, Ph. Flajolet, G. Louchard and G. Schaeffer. Boltzmann samplers for the enumeration of combinatorial structures. *Combin. Probab. Comput.*, 13(4-5):577–625, 2004.

[3] X. G. Viennot. *Heaps of pieces, I: Basic definitions and combinatorial lemmas*, volume 1234/1986, pages 321–350. Springer Berlin/Heidelberg, 1986.