Weakly Directed Walks
A New Class of Self-Avoiding Walks

Axel Bacher (LaBRI), bacher@labri.fr  Mireille Bousquet-Mélou (CNRS, LaBRI), bousquet@labri.fr — LaBRI, Université Bordeaux 1, 351 cours de la Libération, 33405 Talence, France

Introduction & Definitions

Self-avoiding walks (SAW)

**Definition 1**
A lattice walk is *self-avoiding* if it does not visit twice the same vertex.

**Conjecture 2**
The number $c_n$ of SAW of length $n$ and their average end-to-end distance $D_n$ verify:

$$c_n \sim \alpha n^{11/32}\quad \text{and}\quad D_n \sim \kappa n^{1/4}$$

with $\mu \approx 2.64$.

**Motivation:** Find a subclass of SAW both large and conceptually pleasant. Previous record: prudent walks, with $\mu \approx 2.48$.

Weakly directed bridges

**Notation:** A NES-walk (for instance) is a walk that takes only $N$, $E$, and $S$ steps.

**Definition 3**
A walk $v_0 \cdots v_n$ is a *bridge* if every vertex $v \neq v_0$ satisfies $h(v_0) \leq h(v) < h(v_n)$. A nonempty bridge is *irreducible* if it does not factor into two nonempty bridges.

A self-avoiding bridge is weakly directed if every irreducible bridge is either a NES-walk or a NWS-walk (Figure 1).

![Figure 1](image1.png)  
*Figure 1:* A weakly directed bridge factored into five irreducible bridges.

Link with partially directed bridges

**Notation:** Let:

- $B(t)$ be the generating function of NES-bridges;
- $I(t)$ be the generating function of NES-irreducible bridges;
- $W(t)$ be the generating function of weakly directed bridges.

The decomposition into irreducible bridges yields:

$$B(t) = \frac{1}{1 - I(t)}$$

$$W(t) = \frac{1}{1 - (2I(t) - t)}.$$  

Therefore:

$$W(t) = \frac{1}{1 - 2(t^{2} + (2t - 1)G_{k-2} - tG_{k-1})}$$

for $k \geq 2$.

**Theorem 4**
Let $k \geq 0$. The generating function of NES-bridges of height $k$ is

$$B_k(t) = \frac{t^k}{G_k(t)},$$

where $G_k(t)$ is the sequence of polynomials defined by

- $G_0 = 1$;
- $G_k = (1 - t^2 + t^4)G_{k-1} - t^2G_{k-2}$

for $k \geq 2$.

Two different proofs of this result are outlined below (details on demand!).

![Figure 2](image2.png)  
*Figure 2:* First proof of Theorem 4. Let $T_k(t, u)$ be the generating function of walks in a strip of height $k - 1$, with $u$ accounting for the final height. The value of $T_k$ is derived using the kernel method, and $B_k$ follows. [1]

**Enumeration**

**Enumeration of partially directed bridges**

![Figure 3](image3.png)  
*Figure 3:* Second proof of Theorem 4. A NES-bridge of height $k$ is seen as an arbitrary sequence of *generalized steps*, and thus as a walk on a graph with vertices $\{0, \ldots, k - 1\}$. Such walks are enumerated using *heaps of cycles* techniques. [3]

Asymptotics & Random Sampling

**Nature of the series & asymptotics**

**Proposition 5**
The generating functions $B(t)$ and $W(t)$ both have a complex singularity structure and are not $D$-finite.

The number $w_n$ of weakly directed bridges and their average end-to-end distance $E_n$ are asymptotically:

$$w_n \sim \alpha n^\mu\quad \text{and}\quad E_n \sim \kappa n,$$

with $\mu \approx 2.5447$.

The growth constant is larger than for prudent walks, but the average end-to-end distance is still linear.

**Random sampling**

**Proposition 6**
It is possible to sample a weakly directed walk with approximate length $n$ in time $O(n)$, using a Boltzmann sampler. [2]

References

