The spectrum of an asymmetric annihilation process
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Introduction

- Recently Ayyer and Mallick studied an asymmetric annihilation process.
- They determined transfer matrices and the partition function.
- They conjectured the spectrum of this process (including multiplicities).

![Figure 1: Right shift and annihilation in the bulk with rate 1](image1)
![Figure 2: Right shift and annihilation on left boundary with rate α](image2)
![Figure 3: Annihilation on right boundary with rate β](image3)

The transitions in Fig. 1 occur with rate \(1\).
The transitions in Fig. 2 occur with rate \(\alpha\).
The transition in Fig. 3 occurs with rate \(\beta\).

We prove the conjecture by generalizing the original.
We outline a derivation of the partition function in the generalized model, which also reduces to the one obtained by Ayyer and Mallick in the original model.

Theorem and Conjecture (Ayyer and Mallick)

- **Theorem:** The partition function of the system of size \(L\) is given by
  \[ Z_L = 2^{(\alpha^2 + \alpha + 1)/2} (1 + 2\alpha L^{-1}) (1 + \beta L^{-1}) (2\alpha + \beta) \]

- **Conjecture:**
  \[ A_L(\alpha) = \prod_{b \in \mathbb{Z}^L} (\lambda_b - \alpha \delta^{x,y} \cdot b) \]
  where \(\lambda_b = \sum_{e \in \mathbb{Z}^L} (-1)^{b \cdot e} \alpha_e\).
  This implies the original conjecture.

- **Partition function**:
  \[ Z(\alpha, \beta) = \prod_{0 \neq b \in \mathbb{Z}^L} (\lambda_b + \beta \delta^{x,y} \cdot b) \]
  where \(\lambda_b = 2 \sum_{e : b \cdot e = 1} \alpha_e\)

Partition Function for a Model with Inhomogeneities

- The transitions in Fig. 1 at bond \(j\) occur with rate \(\beta_j\).
- The transitions in Fig. 2 occur with rate \(\alpha\).
- The transition in Fig. 3 occurs with rate \(\beta_L\).
- In this case the partition function is
  \[ Z_L(\alpha, \beta) = \prod_{1 \leq j \leq L} (2\alpha + \beta_j) \cdot \prod_{1 \leq i < j \leq L} (\beta_j + \beta_i) \]

Examples of Transition Matrices

\[ M_1 = \begin{pmatrix} -\alpha & \alpha + \beta \\ \alpha & -\alpha - \beta \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 \star 1 \\ \alpha \times \beta \end{pmatrix}, \quad M_3 = \begin{pmatrix} \star 0 1 \alpha 0 1 \star \\ 0 1 0 0 \alpha \star 0 \star 1 0 \star 0 \star 0 \star 0 \star 0 \star 0 \star 0 \star 0 \star \beta 0 1 \star 0 \star 0 \star 0 \star 0 \star 0 \star 0 \star 0 \star 0 \star 0 \star \end{pmatrix} \]

Diagonal elements \(\star\) have to be set such that the column sums vanish.

Inductive Structure of the Transition Matrices

\[ \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and } 1_L \text{ is the identity matrix of size } 2^L. \]

\[ M_L = \begin{pmatrix} L_{11} - \sigma (\Delta \otimes 1_L) & \alpha 1_{L-1} + (\sigma \otimes \Delta) \\ \alpha 1_{L-1} - 1_L - \sigma (\sigma \otimes 1_L) \end{pmatrix} \]

Definitions for Bitvectors \(b = (b_1, \ldots, b_L) \in \mathbb{B}^L\)

- \(\sigma^b = \sigma^{b_1} \otimes \sigma^{b_2} \otimes \ldots \otimes \sigma^{b_L}\)
- \(A_L(\alpha) = \prod_{b \in \mathbb{B}^L} (\lambda_b - \alpha \delta^{x,y} \cdot b)\)
- \(\phi_j : b_1 \ldots b_j \mapsto b_1 \ldots \tilde{b}_j \ldots b_L\)
- \(\psi_j : b \mapsto \phi_j(b) + 1\)
- \(\mathcal{P}_L = \sum_{b \in \mathbb{B}^L} |b| \cdot |\psi_j(b)| (b)\)
- \(B_L(\beta) = \sum_{1 \leq j \leq L} \beta_j \mathcal{P}_j\)
- \(\mathcal{A}_L(\alpha, \beta) = A_L(\alpha) - B_L(\beta)\)
- \(\Delta : b \mapsto b^\Delta = \sum_{1 \leq j \leq L} b_j (b^\Delta)\)

Idea of Proof

- \(H_L\): Hadamard-transform of order \(L\) \(H_L = 2^{-L/2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes 1_L\)
- \(A_L\) diagonalizes under \(H_L\)
- \(B_L\) becomes lower triangular under a suitable reordering of \(H_L\)
- \(H_L \cdot B_L(\beta) \cdot H_L = B_L(\beta^{x,y})\)
- \(H_L\) is \(H_L\) in the basis \(\{b^\Delta, b \in \mathbb{B}^L\}\)

Illustration of the Theorem for \(L = 3, \beta = (\beta, \gamma, \delta)\)

![Illustration](image)

Reference

Arvind Ayyer, Kirone Mallick
Exact results for an asymmetric annihilation process with open boundaries

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