The spectrum of an asymmetric annihilation process

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Introduction

- Recently Ayyer and Mallick studied an asymmetric annihilation process.
- They determined transfer matrices and the partition function.
- They conjectured the spectrum of this process (including multiplicities).

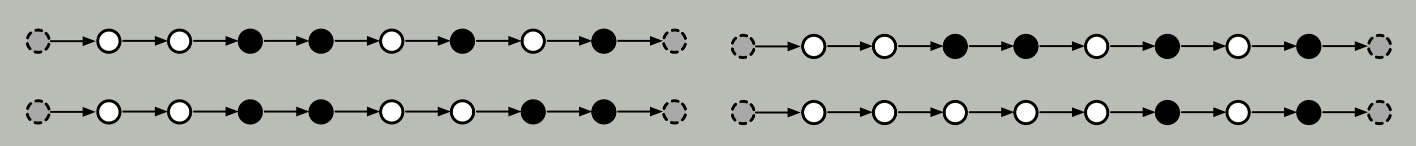

Figure 1: Right shift and annihilation in the bulk with rate 1

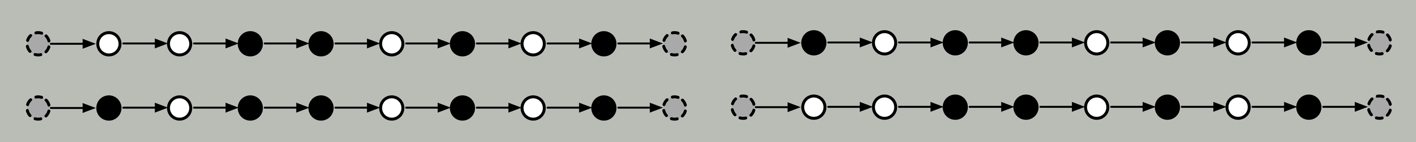

Figure 2: Right shift and annihilation on left boundary with rate α

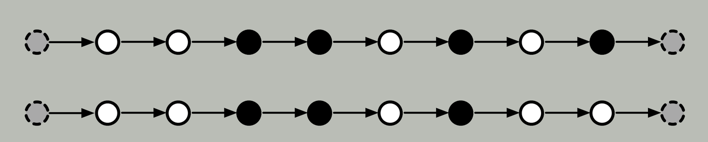

Figure 3: Annihilation on right boundary with rate β

- We prove the conjecture by generalizing the original.
- We outline a derivation of the partition function in the generalized model, which also reduces to the one obtained by Ayyer and Mallick in the original model.

Theorem and Conjecture (Ayyer and Mallick)

- Theorem: The partition function of the system of size L is given by

$$Z_L = 2^{\binom{L-1}{2}}(1+2\alpha)^{L-1}(1+\beta)^{L-1}(2\alpha+\beta).$$

- Conjecture:

$$\det M_L - \lambda 1_L = A_L(\lambda)A_L(\lambda+2\alpha+\beta)B_L(\lambda+\beta)B_L(\lambda+2\alpha),$$

$$A_L(\lambda) = \prod_{k=0}^{\lceil L/2 \rceil} (\lambda+2k)^{\binom{L-1}{2k}}, \quad B_L(\lambda) = \prod_{k=0}^{\lfloor L/2 \rfloor} (\lambda+2k+1)^{\binom{L-1}{2k+1}}.$$

(see reference below)

Main Theorems

- Spectrum:

$$\det \mathcal{M}_L(\alpha,\beta) = \det\left[\mathcal{A}_L(\alpha) - \mathcal{B}_L(\beta)\right] = \prod_{b\in\mathbb{B}^L} \left(\lambda_{b^\Delta} - \beta^{\mathrm{rev}}\cdot b\right)$$

  where $\lambda_b = \sum_{c\in\mathbb{B}^L}(-1)^{b\cdot c}\alpha_c$.
- This implies the original conjecture.
- Partition function:

$$Z(\alpha,\beta) = \prod_{0\neq b\in\mathbb{B}^L} \left(\lambda^*_{b^\Delta} + \beta^{\mathrm{rev}}\cdot b\right)$$

  where $\lambda^*_b = 2\sum_{c:\, b\cdot c=1}\alpha_c$

Partition Function for a Model with Inhomogeneities

- The transitions in FIg. 1 at bond j occur with rate β_j
- The transitions in FIg. 2 occur with rate α
- The transition in FIg. 3 occurs with rate β_L
- In this case the partition function is

$$Z_L(\alpha,\vec{\beta}) = \prod_{1\leq j\leq L}(2\alpha+\beta_j)\cdot\prod_{1\leq i<j\leq L}(\beta_i+\beta_j)$$

Examples of Transition Matrices

$$M_1 = \begin{bmatrix} -\alpha & \alpha+\beta \\ \alpha & -\alpha-\beta \end{bmatrix}, \quad M_2 = \begin{bmatrix} \star & \beta & \alpha & 1 \\ 0 & \star & 1 & \alpha \\ \alpha & 0 & \star & \beta \\ 0 & \alpha & 0 & \star \end{bmatrix}, \quad M_3 = \begin{bmatrix} \star & \beta & 0 & 1 & \alpha & 0 & 1 & 0 \\ 0 & \star & 1 & 0 & 0 & \alpha & 0 & 1 \\ 0 & 0 & \star & \beta & 1 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \star & 0 & 1 & 0 & \alpha \\ \alpha & 0 & 0 & 0 & \star & \beta & 0 & 1 \\ 0 & \alpha & 0 & 0 & 0 & \star & 1 & 0 \\ 0 & 0 & \alpha & 0 & 0 & 0 & \star & \beta \\ 0 & 0 & 0 & \alpha & 0 & 0 & 0 & \star \end{bmatrix}.$$

Diagonal elements $\star$ have to be set such that the column sums vanish.

Inductive Structure of the Transition Matrices

$\sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and 1_L is the identity matrix of size 2^L. Then

$$M_L = \left[\begin{array}{c|c} M_{L-1} - \alpha(\sigma\otimes 1_{L-2}) & \alpha 1_{L-1} + (\sigma\otimes 1_{L-2}) \\ \alpha 1_{L-1} & M_{L-1} - 1_{L-1} - \alpha(\sigma\otimes 1_{L-2}) \end{array}\right],$$

where M_L is a 2×2 block matrix made up of matrices of size 2^{L-1}.

Definitions for Bitvectors $b = (b_1,\ldots,b_L)\in\mathbb{B}^L$

$$\sigma^b = \sigma^{b_1 b_2\ldots b_L} = \sigma^{b_1}\otimes\sigma^{b_2}\otimes\cdots\otimes\sigma^{b_L}$$

$$\mathcal{A}_L(\alpha) = \sum_{b\in\mathbb{B}^L}\alpha_b\sigma^b$$

$$\phi_j : b_1\ldots b_j\ldots b_L \mapsto b_1\ldots\overline{b_j}\ldots b_L$$

$$\psi_j : b\mapsto \phi_j\phi_{j+1}b$$

$$\mathcal{P}_{L,j} = \sum_{b\in\mathbb{B}^L} |\,b\,\rangle\langle\,b\,| - |\,\psi_j^{b_j}(b)\,\rangle\langle\,b\,|$$

$$\mathcal{B}_L(\beta) = \sum_{1\leq j\leq L}\beta_j\mathcal{P}_{L,j}$$

$$\mathcal{M}_L(\alpha,\beta) = \mathcal{A}_L(\alpha) - \mathcal{B}_L(\beta)$$

$$\Delta : b\mapsto b^\Delta = \left[\textstyle\sum_{1\leq i\leq L-j+1} b_i\right]_{1\leq j\leq L}$$

Idea of Proof

- H_L: Hadamard-transform of order L: $H_L = 2^{-L/2}\begin{bmatrix} 1 & 1 \\ 1 & -1\end{bmatrix}^{\otimes L}$
- $\mathcal{A}_L$ diagonalizes under H_L
- $\mathcal{B}_L$ becomes lower triangular under a suitable reordering of H_L

$$\widetilde{H}_L\cdot B_L(\beta)\cdot\widetilde{H}_L = B_L^t(\beta^{\mathrm{rev}})$$

- $\widetilde{H}_L$ is H_L in the basis $\{b^\Delta;\, b\in\mathbb{B}^L\}$

Illustration of the Theorem for $L=3$, $\beta = (\beta,\gamma,\delta)$

b	b^Δ	λ_{b^Δ}	$(\delta,\gamma,\beta)\cdot b$
000	000	$[++++++++]\cdot\alpha$	0
001	100	$[++++----]\cdot\alpha$	β
010	110	$[++----++]\cdot\alpha$	γ
011	010	$[++--++--]\cdot\alpha$	$\beta+\gamma$
100	111	$[+--+-++-]\cdot\alpha$	δ
101	011	$[+--++--+]\cdot\alpha$	$\beta+\delta$
110	001	$[+-+-+-+-]\cdot\alpha$	$\gamma+\delta$
111	101	$[+-+--+-+]\cdot\alpha$	$\beta+\gamma+\delta$

E.g., $b = 101$ contributes the factor

$\alpha_{000}-\alpha_{001}-\alpha_{010}-\alpha_{011}+\alpha_{100}-\alpha_{101}-\alpha_{110}-\alpha_{111}-\beta-\delta.$

Reference

Arvind Ayyer, Kirone Mallick

Exact results for an asymmetric annihilation process with open boundaries

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