# Compositions with Constrained Multiplicities

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## Abstract

*We find the probability that a random composition (ordered partition) of the positive integer $n$ has no parts occurring exactly $j$ times, where $j$ belongs to a specified finite* 'forbidden set' $A$ *of multiplicities.*
*This probability is also studied in the related case of samples $\Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_n)$ of independent, identically distributed random variables with a geometric distribution.*

## 1. Introduction

- We derive generating functions for random compositions of a positive integer $n$ in which no parts occur exactly $j$ times, where $j$ belongs to a specified finite *'forbidden set'* $A$ of multiplicities.
- We refer to such compositions as being *'$A$-avoiding'*.
- We find the probabilities that compositions and samples of geometric random variables are $A$-avoiding.

## 2. Examples

- As a simple example of a forbidden set, we consider a sample where none of the $n$ elements occur exactly $a$ times. I.e., $A = \{a\}$.
- Another example is when a letter can occur only $a$ times or more (or not at all). In this case $A = \{1, 2, \ldots, a-1\}$, for $a \geq 2$.

## 3. Notes

- Note that we do not allow $0$ in the forbidden set.
- We can generalise this by allocating each value a different forbidden set. For example, if the value $2$ is not allowed to occur once, and the number of times that $5$ can occur is anything except $2, 3$ or $6$ times, then we have $A_2 = \{1\}$ and $A_5 = \{2, 3, 6\}$. (We denote the forbidden set for the value $i$ by $A_i$.)

## 4. Compositions

Let $C_{A,d}(x; m)$ be the generating function for the number of $A$-avoiding compositions of $n$ with exactly $m$ parts from the set $[d] = \{1, 2, \ldots, d\}$. We have

$$\frac{C_{A,d}(x;m)}{m!} = \sum_{\substack{j=0 \\ j \notin A}}^{m} \frac{x^{dj}}{j!} \frac{C_{A,d-1}(x; m-j)}{(m-j)!}.$$

Solving this recurrence gives

**Proposition 1** The generating function $C_A(x, y) = \sum_{m \geq 0} C_A(x; m) \frac{y^m}{m!}$ is given by

$$C_A(x, y) = \prod_{k \geq 1} \left( e^{x^k y} - \sum_{j \in A} \frac{x^{kj} y^j}{j!} \right),$$

where $C_A(x; m)$ is the generating function for the number of $A$-avoiding compositions of $n$ with exactly $m$ parts in $\mathbb{N}$.

Let $C_A(n, m)$ be the number of $A$-avoiding compositions of $n$ with $m$ parts and $C_A(n) = \sum_{m \geq 1} C_A(n, m)$ be the number of $A$-avoiding compositions of $n$.

**Corollary 1** The generating function $C_A(x) = \sum_{n \geq 0} C_A(n) x^n$ is given by

$$C_A(x) = \int_0^\infty e^{-y} \prod_{k \geq 1} \left( e^{x^k y} - \sum_{j \in A} \frac{x^{kj} y^j}{j!} \right) dy.$$

**Theorem 1** Fix $a \in \mathbb{N}$. Let $A_i = \{a\}$ for all $i = 1, 2, \ldots, \ell$ and $A_{\ell+i} = \emptyset$ for all $i \geq 1$. The ordinary generating function for the number of compositions $\pi$ of $n$ such that $\pi$ does not contain part $i$ exactly $a$ times for all $i = 1, 2, \ldots, \ell$ is given by

$$\sum_{m \geq 0} D_{A_1, A_2, \ldots}(x; m) = \frac{1-x}{1-2x} + \sum_{j=1}^{\ell} \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq \ell} (-1)^j \frac{j! x^{a \sum_{k=1}^{j} i_k}}{\left(1 - \frac{x}{1-x} + \sum_{k=1}^{j} x^{a i_k}\right)^{j+1}}.$$

Even in this simple case of $A = \{a\}$ it does not seem easy to find asymptotic estimates for the coefficients from the generating functions appearing above. Instead we use the probabilistic argument as given in [1, 2] to explain the relationship between compositions of $n$ and the special case for geometric random variables when $p = 1/2$.

Asymptotic estimates are consequently derived for compositions by equipping the set of all compositions of $n$ with the uniform probability measure and considering the probability that a randomly chosen composition of $n$ is $A$-avoiding.

## 5. Geometric random variables

Let $\Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_n)$ be a sample of independent identically distributed (i.i.d.) geometric random variables with parameter $p$, that is, $\mathbb{P}(\Gamma_i = k) = pq^{k-1}$, with $p + q = 1$, where $k = 1, 2, \ldots$ and $i = 1, 2, \ldots, n$.

The method used in [3] can be applied to the problem described above. We use a recursion on the probabilities and then use Poissonisation to enable us to use Mellin transforms and then de-Poissonisation to obtain our asymptotic estimates.

**Theorem 2** Let $A$ be any finite set of positive integers. The probability $p_n$ that a geometric sample of length $n$ has no letter appearing with multiplicity $j$, for any $j \in A$ is (asymptotically as $n \to \infty$)

$$p_n = 1 - \frac{T^*(0)}{\ln(1/q)} - \delta\big(\log_{1/q}(n/q)\big) + O(n^{-1}),$$

with

$$T^*(0) = \sum_{j \in A} p^j \sum_{n \geq 0} p_n q^n \frac{1}{n+j} \binom{n+j}{j}$$

and

$$\delta(x) = \frac{1}{\ln(1/q)} \sum_{k \neq 0} T^*(\chi_k) e^{-2k\pi i x}$$

where

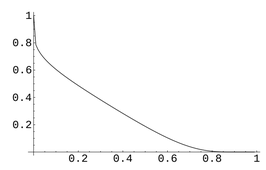
$$T^*(\chi_k) = \sum_{j \in A} \frac{p^j}{j!} \sum_{n \geq 0} p_n \frac{q^n}{n!} \Gamma(n + j + \chi_k), \quad \text{for} \quad k \in \mathbb{Z} \backslash \{0\},$$

for $\chi_k := \frac{2k\pi i}{\ln(1/q)}$. Here $\delta(x)$ is a periodic function of $x$ with period $1$, mean $0$ and small amplitude.

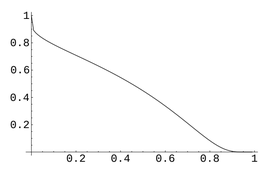
For example if we want the probability that no element occurs exactly once, we have a main term for $p_n$ of

$$1 - \frac{p}{\ln(1/q)} \sum_{n \geq 0} p_n q^n.$$

The main term is plotted as a function of $p$ below, for this and another case.




*No element occurs exactly once*




*No element occurs exactly twice*

In spite of what the graphs tend to suggest for $q$ near $1$, the main term here is strictly greater than zero for every $0 < p < 1$ as

$$T^*(0) = p^a \sum_{n \geq 0} p_n q^n \frac{1}{n+a} \binom{n+a}{a} \leq p^a \sum_{n \geq 0} q^n \frac{1}{n+a} \binom{n+a}{a} = p^a \frac{(1-q)^{-a}}{a}$$
$$\leq 1 < \ln(1/q).$$

## 6. Compositions revisited

**Corollary 2** Let $A$ be any finite set of positive integers. The probability $p_n$ that a composition of $n$ has no part appearing with multiplicity $j$, for any $j \in A$ is (asymptotically as $n \to \infty$)

$$p_n = 1 - \frac{T^*(0)}{\ln 2} - \delta(\log_2 n) + O\left(\frac{\ln^{3/2} n}{\sqrt{n}}\right),$$

with

$$T^*(0) = \sum_{j \in A} \left(\frac{1}{2}\right)^j \sum_{n \geq 0} p_n \left(\frac{1}{2}\right)^n \frac{1}{n+j} \binom{n+j}{j}, \quad \text{and where} \quad \delta(x) = \frac{1}{\ln 2} \sum_{k \neq 0} T^*(\chi_k) e^{-2k\pi i x}$$

is a periodic function of $x$ with period $1$, mean $0$ and small amplitude, with $\chi_k = \frac{2k\pi i}{\ln 2}$ and

$$T^*(\chi_k) = \sum_{j \in A} \frac{1}{j!} \left(\frac{1}{2}\right)^j \sum_{n \geq 0} \frac{p_n}{n!} \left(\frac{1}{2}\right)^n \Gamma(n + j + \chi_k), \quad \text{for} \quad k \in \mathbb{Z} \backslash \{0\}.$$

## References

[1] P. Hitczenko and C. D. Savage. On the multiplicity of parts in a random composition of a large integer. *SIAM J. Discrete Math.*, 18:418–435, 2004.

[2] P. Hitczenko and G. Louchard. Distinctness of compositions of an integer: a probabilistic analysis. *Random Struct. Alg.*, 19:407–437, 2001.

[3] P. Hitczenko and A. Knopfmacher. Gap-free compositions and gap-free samples of geometric random variables. *Discrete Math.*, 294:225–239, 2005.

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