Compositions with Constrained Multiplicities

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1. Introduction

We find the probability that a random composition (ordered partition) of the positive integer \( n \) has no parts occurring exactly \( j \) times, where \( j \) belongs to a specified finite forbidden set \( A \) of multiplicities. This probability is also studied in the related case of samples \( \Gamma = \{T_1, T_2, \ldots, T_n\} \) of independent, identically distributed random variables with a geometric distribution.

We derive generating functions for random compositions of a positive integer \( n \) in which no parts occur exactly \( j \) times, where \( j \) belongs to a specified finite forbidden set \( A \) of multiplicities. We refer to such compositions as being \( \Lambda \)-avoiding.

We find the probabilities that compositions and samples of geometric random variables are \( \Lambda \)-avoiding.

2. Examples

As a simple example of a forbidden set, we consider a sample where none of the \( n \) elements occur exactly \( a \) times. i.e., \( A = \{a\} \),

Another example is when a letter can occur only \( a \) times or more (or not at all). In this case \( A = \{1, 2, \ldots, a - 1\} \), for \( a \geq 2 \).

3. Notes

Note that we do not allow \( 0 \) in the forbidden set.

We can generalise this by allocating each value a different forbidden set. For example, if the value 2 is not allowed to occur once, and the number of times that 5 can occur is anything except 2, 3 or 6 times, then we have \( A_2 = \{1\} \) and \( A_5 = \{1, 3, 5\} \). (We denote the forbidden set for the value \( i \) by \( A_i \)).

4. Compositions

Let \( C_{A_1}(z;m) \) be the generating function for the number of \( A_1 \)-avoiding compositions of \( n \) with exactly \( m \) parts from the set \( \{1, 2, \ldots, d\} \). We have

\[
C_{A_1}(z;m) = \sum_{k=1}^{\infty} \frac{z^k C_{A_1}(z;m-k)}{|m-k|!}.
\]

Solving this recurrence gives

Proposition 1 The generating function \( C_{A_1}(z) = \sum_{n=0}^{\infty} C_{A_1}(z)n^n \) is given by

\[
C_{A_1}(z) = \frac{1 - e^{-z}}{1 - e^{-z} \sum_{f \in A_1} \frac{e^{zf}}{z^n} f^n}.
\]

where \( C_{A_1}(z;m) \) is the generating function for the number of \( A_1 \)-avoiding compositions of \( n \) with exactly \( m \) parts in \( A_1 \).

Let \( C_{A_1}(n,m) \) be the number of \( A_1 \)-avoiding compositions of \( n \) with \( m \) parts and \( C_{A_1}(n) = \sum_{m=1}^{\infty} C_{A_1}(n,m) \) be the number of \( A_1 \)-avoiding compositions of \( n \).

Corollary 1 The generating function \( C_{A_1}(z) = \sum_{n=0}^{\infty} C_{A_1}(z)n^n \) is given by

\[
C_{A_1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{i=1}^{n} \left( \frac{e^{zi}}{i} \right)^{-A_1} \delta_{n,j}.
\]

Theorem 2 Let \( A \) be any finite set of positive integers. The probability \( p_A(n) \), that a geometric sample of length \( n \) has no letter appearing with multiplicity \( j \), for any \( j \in A \) is (asymptotically as \( n \to \infty \))

\[
p_A(n) = 1 - \frac{\Gamma(n+1)}{\Gamma(n+1) + \Gamma(n+1 - j)}.
\]

Even in this simple case of \( A = \{a\} \) it does not seem easy to find asymptotic estimates for the coefficients from the generating functions appearing above. Instead we use the probabilistic argument as given in [1, 2] to explain the relationship between compositions of \( n \) and the special case for geometric random variables when \( p = 1/2 \).

Asymptotic estimates are consequently derived for compositions by equipping the set of all compositions of \( n \) with the uniform probability measure and considering the probability that a randomly chosen composition of \( n \) is \( \Lambda \)-avoiding.

5. Geometric random variables

Let \( \Gamma = \{T_1, T_2, \ldots, T_n\} \) be a sample of independent identically distributed (i.i.d.) geometric random variables with parameter \( p \), that is, \( P(T_i = k) = p^k \cdot q \), with \( p + q = 1 \), where \( k = 1, 2, \ldots, n \).

The method used in [3] can be applied to the problem described above. We use a recursion on the probabilities and then use Poissonisation to enable us to use Mellin transforms and then de-Poissonisation to obtain our asymptotic estimates.

Theorem 2 Let \( A \) be any finite set of positive integers. The probability \( p_A(n) \), that a geometric sample of length \( n \) has no letter appearing with multiplicity \( j \), for any \( j \in A \) is (asymptotically as \( n \to \infty \))

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Corollary 2 Let \( A \) be any finite set of positive integers. The probability \( p_A(n) \), that a composition of \( n \) has no part appearing with multiplicity \( j \), for any \( j \in A \) is (asymptotically as \( n \to \infty \))

\[
p_A(n) = 1 - \frac{\Gamma(n+1)}{\Gamma(n+1) + \Gamma(n+1 - j)}.
\]

with

\[
p_A(n) = \frac{1}{\Gamma(n+1)} \sum_{m=0}^{n} \frac{\Gamma(m+1)}{\Gamma(m+1 - j)} (\frac{1}{n} + \frac{1}{j} + \cdots + \frac{1}{j^{m-1}}) \delta_{m,j}.
\]

where \( \delta_{m,j} \) is a periodic function of \( z \) with period 1, mean 0 and small amplitude, with \( \delta_{m,j} = \frac{\delta_{m,j}}{\Gamma(n+1)} \) and

\[
\Gamma(n+1) = \frac{1}{\Gamma(n+1)} \sum_{m=0}^{n} \frac{\Gamma(m+1)}{\Gamma(m+1 - j)} (\frac{1}{n} + \frac{1}{j} + \cdots + \frac{1}{j^{m-1}}) \delta_{m,j}.
\]

for \( k \in Z(0) \), for \( x_k \neq \frac{\delta_{m,j}}{\Gamma(n+1)} \).

For example if we want the probability that no element occurs exactly once, we have a main term of

\[
1 - p \frac{\delta_{m,j}}{\Gamma(n+1)} \delta_{m,j}.
\]

The main term is plotted as a function of \( p \) below, for this and another case.

\[\text{No element occurs exactly once}\]

\[\text{No element occurs exactly twice}\]

In spite of what the graphs tend to suggest for \( p \) near 1, the main term here is strictly greater than zero for every \( 0 < p < 1 \) as

\[
\Gamma(n+1) = \frac{1}{\Gamma(n+1)} \sum_{m=0}^{n} \frac{\Gamma(m+1)}{\Gamma(m+1 - j)} (\frac{1}{n} + \frac{1}{j} + \cdots + \frac{1}{j^{m-1}}) \delta_{m,j}.
\]

References

